MATH 247 Calculus III, Final Definitions and Theorems

Proposition (9.4). *$A \subseteq \mathbb{R}^n$, $f\colon A \to \mathbb{R}$, $\vec{a} \in \text{int}(A)$. Let $\vec{v} \neq \vec{0} \in \mathbb{R}^n$. Suppose that $(\partial_{\vec{v}} f)(\vec{a})$ exists. Then for every $\alpha \in \mathbb{R}$, the directional derivative $(\partial_{\alpha\vec{v}} f)(\vec{a})$, and*

$$(\partial_{\alpha\vec{v}} f)(\vec{a}) = \alpha(\partial_{\vec{v}} f)(\vec{a})$$

Proof. If $\alpha = 0$, then the equation becomes $0 = 0$.

Assume $\alpha \neq 0$. Denote $\alpha\vec{v} = \vec{w}$. Then

$$\begin{aligned}
&\lim_{t\to 0} \frac{f(\vec{a}+t\vec{w}) - f(\vec{a})}{t} \\
=&\lim_{t\to 0} \frac{f(\vec{a}+t\alpha\vec{v}) - f(\vec{a})}{t\alpha}\alpha \\
=&\lim_{s\to 0} \frac{f(\vec{a}+s\vec{v}) - f(\vec{a})}{s}\alpha \\
=&(\partial_{\vec{v}} f)(\vec{a}) \cdot \alpha
\end{aligned}$$

□

Proposition (9.8). *$A \subseteq \mathbb{R}^n$, $f\colon A \to \mathbb{R}$, $\vec{a} \in \text{int}(A)$. Suppose $(\partial_{\vec{v}} f)(\vec{a})$ exists for all $\vec{v} \in bR^n$, and that we have (Add) property. Then for every $\vec{v} \in \mathbb{R}^n$, we have*

$$(\partial_{\vec{v}} f)(\vec{a}) = \langle \vec{v}\,,\, (\nabla f)(\vec{a})\,\rangle$$

Proof. Fix $\vec{v} = (v^{(1)}, \cdots, v^{(n)}) \in \mathbb{R}^n$, and write $\vec{v} = v^{(1)}\vec{e}_1 + \cdots + v^{(n)}\vec{e}_n$. Then

$$\begin{aligned}
(\partial_{\vec{v}} f)(\vec{a}) &= (\partial_{v^{(1)}\vec{e}_1+\cdots+v^{(n)}\vec{e}_n} f)(\vec{a}) \\
&= (\partial_{v^{(1)}\vec{e}_1} f)(\vec{a}) + \cdots + (\partial_{v^{(n)}\vec{e}_n} f)(\vec{a}) \\
&= v^{(1)}(\partial_1 f)(\vec{a}) + \cdots + v^{(1)}(\partial_n f)(\vec{a}) \\
&= \langle \vec{v}\,,\, (\nabla f)(\vec{a})\,\rangle
\end{aligned}$$

□

Proposition (10.2). *$A \in \mathbb{R}^n$, $f\colon A \to \mathbb{R}$, $\vec{a} \in \text{int}(A)$, $\vec{v} \in \mathbb{R}^n$ and suppose that $(\partial_{\vec{v}} f)(\vec{a})$ exists. If $\vec{a}$ is a point of local extremum for f, then $(\partial_{\vec{v}} f)(\vec{a}) = 0$.*

Proof. Assume that $\vec{a}$ is a local minimum. Let $r > 0$ be such that $\text{B}(\vec{a}, r) \subseteq A$ and $f(\vec{a}) \leq f(\vec{x})$ for all $\vec{x} \in \text{B}(\vec{a}, r)$. Let $c = \frac{r}{1+||\,\vec{v}\,||}$. Then have $\vec{a} + t\vec{v} \in \text{B}(\vec{a}, r)$, $\forall t \in (-c, c)$. Define $h\colon (-c, c) \to \mathbb{R}$

$$h(t) = f(\vec{a} + t\vec{v})$$

For every $t \in I$, we have $h(t) = f(\vec{a} + t\vec{v}) \geq f(\vec{a}) = h(0)$. Hence 0 is a point of minimum for h on I.

Then we have

$$\lim_{t\to 0} \frac{h(t) - h(0)}{t - 0} = \lim_{t\to 0} \frac{f(\vec{a}+t\vec{v}) - f(\vec{a})}{t} = (\partial_{\vec{v}} f)(\vec{a})$$

Since $(\partial_{\vec{v}} f)(\vec{a})$ exists, then the left hand side must exist as well. Then $h'(0) = (\partial_{\vec{v}} f)(\vec{a})$. Since $h'(0)$ exists and 0 is a point of minimum, then $h'(0) = 0$. So $(\partial_{\vec{v}} f)(\vec{a}) = 0$, as required. □

Theorem (10.5). *$A \subseteq \mathbb{R}^n$ open. $f\colon A \to \mathbb{R}$. $\vec{v} \neq \vec{0} \in \mathbb{R}^n$. Suppose that $(\partial_{\vec{v}} f)(\vec{a})$ exists at every $\vec{a} \in A$. Let $\vec{x}, \vec{y} \in A$ be such that $\text{Co}(\vec{x}, \vec{y}) \subseteq A$ and such that $\vec{y} - \vec{x} = \alpha\vec{v}$ for some $\alpha \in \mathbb{R}$. Then $\exists \vec{b} \in \text{Co}(\vec{x}, \vec{y})$ such that*

$$f(\vec{y}) - f(\vec{x}) = \alpha \cdot (\partial_{\vec{v}} f)(\vec{b}) = (\partial_{\alpha\vec{v}} f)(\vec{b})$$

Proof. Will assume $\alpha \neq 0$. Define $\varphi(t) = f((1-t)\vec{x} + t\vec{y})$. Idea is to use MVT from calculus I on φ.

Claim 1.

(a). $\varphi(0) = f(\vec{x}), \varphi(1) = f(\vec{y})$.

(b). For every $t \in [0, 1]$, φ is differentiable at t with $\varphi'(t) = \alpha(\partial_{\vec{v}} f)((1-t)\vec{x} + t\vec{y})$.

Verif of Claim 1.

(a) .....

(b) Fix $t_0 \in [0, 1]$ where we check the differentiability of φ. Denote $(1 - t_0)\vec{x} + t_0\vec{y} = \vec{a} \in A$. Look at the newton quotient $\frac{\varphi(t_0+h)-\varphi(t_0)}{h}$, with $h \neq 0$ such that $t_0 + h \in [0, 1]$. Then we have $\varphi(t_0) = f(\vec{a})$. $\varphi(t_0 + h) = f(\vec{a} - h\vec{x} + h\vec{y}) = f(\vec{a} + h\alpha\vec{v})$. So

$$\lim_{h\to 0} \frac{\varphi(t_0+h) - \varphi(t_0)}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{f(\vec{a}+h\alpha\vec{v})-f(\vec{a})}{h\alpha}\cdot\alpha \\
&=\alpha\cdot(\partial_{\vec{v}}f)(\vec{a})
\end{aligned}
$$

Hence $\varphi'(t_0)$ exists and has the claimed formula.

Claim 2. $\exists c\in(0,1)$ such that $\varphi(1)-\varphi(0)=\varphi'(c)$.

Verif of Claim 2.

φ is continuous at every $t\in[0,1]$, since φ is differentiable at every $t\in[0,1]$.

Claim 3. $\exists\vec{b}\in\mathrm{Co}(\vec{x},\vec{y})$ such that $f(\vec{y})-f(\vec{x})=\alpha\cdot(\partial_{\vec{v}}f)(\vec{b})$.

Verif of claim 3.

Take $c\in(0,1)$ as in claim 2. put $\vec{b}=(1-c)\vec{x}+c\vec{y}$. Then $f(\vec{y})-f(\vec{x})=\varphi(1)-\varphi(0)=\varphi'(c)=\alpha\cdot(\partial_{\vec{v}}f)(\vec{b})$. □

Theorem (11.3)**.** $A\subseteq\mathbb{R}^n$, $f\in C^1(A,\mathbb{R})$. *Then for every* $\vec{a}\in A$ *we have*

$$
(\mathrm{L-Approx}) \qquad \lim_{\vec{x}\to\vec{a}} \frac{|f(\vec{x})-f(\vec{a})-\langle\vec{x}-\vec{a}\,,\,(\nabla f)(\vec{a})\,\rangle|}{||\,\vec{x}-\vec{a}\,||}=0
$$

Corollary (11.4)**.** $A\subseteq\mathbb{R}^n$ *open,* $f\in C^1(A,\mathbb{R})$, $\vec{a}\in A$. *Then for every* $\vec{v}\in\mathbb{R}^n$, *the directional derivative* $(\partial_{\vec{v}}f)(\vec{a})$ *exists, and have* $(\partial_{\vec{v}}f)(\vec{a})=\langle\vec{v}\,,\,(\nabla f)(\vec{a})\,\rangle$.

Proof. In (L-Approx) we pick $\vec{x}$ of the form $\vec{a}+t\vec{v}$. Then $\vec{x}\to\vec{a}$ be comes $t\to 0$.

Then multiply the limit by $||\,\vec{v}\,||$

$$
\lim_{t\to 0}\frac{|f(\vec{a}+t\vec{v})-f(\vec{a})-\langle(\vec{a}+t\vec{v})-\vec{a}\,,\,(\nabla f)(\vec{a})\,\rangle|}{||\,\vec{a}+t\vec{v}-\vec{a}\,||}\cdot||\,\vec{v}\,||=0||\,\vec{v}\,||=0
$$

$$
\lim_{t\to 0}\left|\frac{f(\vec{a}+t\vec{v})-f(\vec{a})-t\langle\vec{v}\,,\,(\nabla f)(\vec{a})\,\rangle}{t}\right|=0
$$

$$
\lim_{t\to 0}\left|\frac{f(\vec{a}+t\vec{v})-f(\vec{a})}{a}-\langle\vec{v}\,,\,(\nabla f)(\vec{a})\,\rangle\right|=0
$$

$$
\lim_{t\to 0}\frac{f(\vec{a}+t\vec{v})-f(\vec{a})}{a}=\langle\vec{v}\,,\,(\nabla f)(\vec{a})\,\rangle
$$

□

Corollary (11.5)**.** $A\subseteq\mathbb{R}^n$ *open,* $f\in C^1(A,\mathbb{R})$, $\vec{a}\in A$. *The directional derivatives at* $\vec{a}$ *have (Add) property.*

Lemma (11.6)**.** $A\subseteq\mathbb{R}^n$ *open,* $f\in C^1(A,\mathbb{R})$, $\vec{a}\in\mathbb{R}$. *Pick* $r>0$ *such that* $\mathrm{B}(\vec{a},r)\subseteq A$. *Then for every* $\vec{x}\in\mathrm{B}(\vec{a},r)$ *we can find* $\vec{b}_1,\cdots\vec{b}_n\in\mathrm{B}(\vec{a},r)$ *such that* $f(\vec{x})-f(\vec{a})=\langle\vec{x}-\vec{a}\,,\,\vec{w}\,\rangle$ *with* $\vec{w}=((\partial_1 f)(\vec{b}_1),\cdots,(\partial_n f)(\vec{b}_n))$.

Proof. Fix $\vec{x}\in\mathrm{B}(\vec{a},r)$. Consider vectors $\vec{x}_0,\vec{x}_1,\cdots,\vec{x}_n$, defined as follows:

$$
\vec{x}_0=\vec{a}=(a^{(1)},\cdots,a^{(n)})
$$

$$
\vec{x}_1=\vec{a}=(x^{(1)},\cdots,a^{(n)})
$$

$$
\vec{x}_2=\vec{a}=(x^{(1)},x^{(2)},\cdots,a^{(n)})
$$

$$
\cdots
$$

$$
\vec{x}_n=\vec{a}=(x^{(1)},\cdots,x^{(n)})=\vec{x}
$$

Note that for every $1\le i\le n$ we have $||\,\vec{x}_i-\vec{a}\,||\le||\,\vec{x}-\vec{a}\,||<r$. Hence $\vec{x}_0,\vec{x}_1,\cdots,\vec{x}_n\in\mathrm{B}(\vec{a},r)\subseteq A$.

Claim for every $1\le i\le n$ there exists $\vec{b}_i\in\mathrm{Co}(\vec{x}_{i-1},\vec{x}_i)$ such that

$$
f(\vec{x}_i)-f(\vec{x}_i)-f(\vec{x}_{i-1})=(x^{(i)}-a^{(i)})(\partial_i f)(\vec{b}_i)
$$

Verification of the claim.

$$
\begin{aligned}
&\vec{x}_i-\vec{x}_{i-1} \\
=&(x^{(i)}-a^{(i)})\cdot\vec{e}_i \\
=&\alpha\vec{e}_i
\end{aligned}
$$

Apply MVT in direction $\vec{e}_i$ with endpoints $\vec{x}_{i-1}$ and $\vec{x}_i$, then $\exists\vec{b}_i\in\mathrm{Co}(\vec{x}_{i-1},\vec{x}_i)$ such that $f(\vec{x}_i)-f(\vec{x}_{i-1})=(x^{(i)}-a^{(i)})(\partial_i f)(\vec{b}_i)$. Done with claim.

Then

$$\begin{aligned} f(\vec{x}) - f(\vec{a}) &= f(\vec{x}_m) - f(\vec{x}_0) \\ &= f(\vec{x}_m) - f(\vec{x}_{m-1}) + \cdots + f(\vec{x}_1) - f(\vec{x}_0) \\ &= \sum_{i=1}^{m} f(\vec{x}_i) - f(\vec{x}_{i-1}) \\ &= \sum_{i=1}^{m} (x^{(i)} - a^{(i)})(\partial_i f)(\vec{b}_i) \\ &= \langle \vec{x} - \vec{a} \, , \, \vec{w} \, \rangle \end{aligned}$$

where $\vec{w} = ((\partial_1 f)(\vec{b}_1), \cdots, (\partial_n f)(\vec{b}_n))$. ☐

Proof of Theorem 11.3. Given $\epsilon > 0$, we want to find $\delta > 0$ such that $\mathrm{B}(\vec{a}, \delta) \subseteq A$ and such that

$$\text{(Want)} \qquad \frac{|f(\vec{x}) - f(\vec{a}) - \langle \vec{x} - \vec{a} \, , \, (\nabla f)(\vec{a}) \, \rangle|}{|| \, \vec{x} - \vec{a} \, ||} < \epsilon$$

for all $\vec{x} \in \mathrm{B}(\vec{a}, \delta) \setminus \{\vec{a}\}$.

Fix $r_0 > 0$ such that $\mathrm{B}(\vec{a}, r_0) \subseteq A$. For every $1 \leq i \leq n$, we know that $\partial_i f$ is continuous at $\vec{a}$ hence $\exists 0 \leq r_i \leq r_0$ such that for all $\vec{y} \in \mathrm{B}(\vec{a}, r_i)$ we have

$$|(\partial_i f)(\vec{y}) - (\partial_i f)(\vec{a})| < \frac{\epsilon}{n}$$

Put $\delta = \min(r_1, \cdots, r_n)$. Claim δ is good for (Want).

Verification of claim. Pick $\vec{x} \in \mathrm{B}(\vec{a}, \delta) \setminus \{a\}$ for which we prove that

$$\text{(Want}'\text{)} \qquad |f(\vec{x}) - f(\vec{a}| - \langle \vec{x} - \vec{a} \, , \, (\nabla f)(\vec{a}) \, \rangle < \epsilon || \, \vec{x} - \vec{a} \, ||.$$

Lemma 11.6 gives us points $\vec{b}_1, \cdots, \vec{b}_n \in \mathrm{B}(\vec{a}, \delta)$ such that

$$f(\vec{x} - \vec{a}) = \langle \vec{x} - \vec{a} \, , \, \vec{w} \, \rangle$$

where $\vec{w} = ((\partial_1 f)(\vec{b}_1), \cdots, (\partial_n f)(\vec{b}_n))$.

Then

$$\begin{aligned} & |f(\vec{x}) - f(\vec{a}| - \langle \vec{x} - \vec{a} \, , \, (\nabla f)(\vec{a}) \, \rangle \\ = & |\langle \vec{x} - \vec{a} \, , \, \vec{w} \, \rangle - \langle \vec{x} - \vec{a} \, , \, (\nabla f)(\vec{a}) \, \rangle| \\ = & |\langle \vec{x} - \vec{a} \, , \, \vec{w} - (\nabla f)(\vec{a}) \, \rangle| \\ \leq & || \, \vec{x} - \vec{a} \, || \cdot || \, \vec{w} - (\nabla f)(\vec{a}) \, || \\ \leq & || \, \vec{x} - \vec{a} \, || \cdot || \, \vec{w} - (\nabla f)(\vec{a}) \, ||_1 \\ = & || \, \vec{x} - \vec{a} \, || \cdot \sum_{i=1}^{m} \left| (\partial_i f)(\vec{b}_i) - (\partial_i f)(\vec{a}) \right| \\ < & || \, \vec{x} - \vec{a} \, || \cdot \sum_{i=1}^{m} \frac{\epsilon}{n} \\ = & \epsilon \cdot || \, \vec{x} - \vec{a} \, || \end{aligned}$$

☐

Theorem (13.2)**.** *$A \subseteq \mathbb{R}^n$ open, $f \in C^1(A, \mathbb{R})$. Let $I \subseteq \mathbb{R}$ be an open interval and let $\gamma \colon I \to \mathbb{R}^n$ be a differentiable path such that $\gamma(t) \in A$ for all $t \in I$. Define $u \colon I \to \mathbb{R}$ by $u(t) = f(\gamma(t))$ Then u is differentiable with*

$$u'(t) = \langle (\nabla f)(\gamma(t)) \, , \, \gamma'(t) \, \rangle$$

Proof. Fix $t_0 \in I$ for which we will prove that the Chain Rule holds. So we need

$$\lim_{t \to t_0} \frac{u(t) - u(t_0)}{t - t_0} = \langle (\nabla f)(\gamma(t_0)) \, , \, \gamma'(t_0) \, \rangle$$

We will do this limit by sequence. Let $(t_k)_{k=1}^{\infty}$ in I such that $t_k \to t_0$. Will show that

$$\lim_{k\to\infty} \frac{u(t_k) - u(t_0)}{t_k - t_0} = \langle (\nabla f)(\gamma(t_0)) \, , \, \gamma'(t_0) \rangle$$

Denote $\gamma(t_0) = \vec{a} \in A$, $\gamma(t_k) = \vec{t_k} \in A, \forall k \in \mathbb{N}$. Then $(\vec{x}_k)_{k=1}^{\infty}$ is a sequence in A.
Claim 1. We have $\vec{x}_k \to \vec{a}$, and moreover that

$$\lim_{k\to\infty} \frac{1}{t_k - t_0}(\vec{x}_k - \vec{a}) = \gamma'(t_0)$$

Verif of Claim 1. For every $k \in \mathbb{N}$ we have

$$\vec{x}_k = \gamma(t_k) = (\gamma^{(1)}(t_k), \cdots, \gamma^{(n)}(t_k))$$

where $\gamma^{(1)}, \cdots \gamma^{(n)} \colon I \to \mathbb{R}$ are differentiable, hence continuous.
When $k \to \infty$, get $\gamma^{(i)}(t_k) \to \gamma^{(i)}(t_0)$. So $\vec{x}_k \to (\gamma^{(1)}(t_0), \cdots, \gamma^{(n)}(t_0))$. Hence $\vec{x}_k \to \vec{a}$ as needed.
Moreover,

$$\frac{1}{t_k - t_0}(\vec{x}_k - \vec{a}) = (\frac{\gamma^{(1)}(t_k) - \gamma^{(1)}(t_0)}{t_k - t_0}, \cdots, \frac{\gamma^{(n)}(t_k) - \gamma^{(n)}(t_0)}{t_k - t_0}) \to ((\gamma^{(1)})'(t_0), \cdots, (\gamma^{(n)})'(t_0)) = \gamma'(t_0)$$

Claim 2. Pick $r > 0$ such that $\mathrm{B}(\vec{a}, r) \subseteq A$, and pick $k_0 \in \mathbb{N}$ such that $\vec{x}_k \in \mathrm{B}(\vec{a}, r)$ for all $k \geq k_0$. Then for every $k \geq k_0$ have $\mathrm{Co}(\vec{a}, \vec{x}_k) \subseteq A$, and we can find $\vec{b}_k \in \mathrm{Co}(\vec{a}, \vec{x}_k)$ such that

$$\frac{u(t_k) - u(t_0)}{t_k - t_0} = \langle (\nabla f)(\vec{b}_k) \, , \, \frac{1}{t_k - t_0}(\vec{x}_k - \vec{a}) \rangle$$

Verif of Claim 2. Application of MVT.
Claim 3. Let $(\vec{b}_k)_{k=0}^{\infty}$ be as in Claim 2. Then $\vec{v}_k \to \vec{a}$, and therefore $(\nabla f)(\vec{b}_k) \to (\nabla f)(\vec{a})$.
Verif of Claim 3. For every $k \geq k_0$ we have $\vec{b}_k \in \mathrm{Co}(\vec{a}, \vec{x}_k)$, have $||\,\vec{b}_k - \vec{a}\,|| \leq ||\,\vec{x}_k - \vec{a}\,||$. By squeeze theorem $\vec{b}_k \to \vec{a}$. Then for every $1 \leq i \leq n$ get $(\partial_i f)(\vec{b}_k) \to (\partial_i f)(\vec{a})$ because $\partial_i f$ is continuous on A. Then $(\nabla f)(\vec{b}_k) \to (\nabla f)(\vec{a})$.
Claim 4. We have

$$\lim_{k\to\infty} \frac{u(t_k) - u(t_0)}{t_k - t_0} = \langle (\nabla f)(\gamma(t_0)) \, , \, \gamma'(t_0) \rangle$$

Verif of Claim 4. Have $(\nabla f)(\vec{b}_k) \to (\nabla f)(\vec{a})$

$$\lim_{k\to\infty} \frac{1}{t_k - t_0}(\vec{x}_k - \vec{a}) = \gamma'(t_0)$$

So

$$\langle (\nabla f)(\vec{b}_k) \, , \, \frac{1}{t_k - t_0}(\vec{x}_k - \vec{a}) \rangle \to \langle (\nabla f)(\vec{a}) \, , \, \gamma'(t) \rangle$$

Since

$$\frac{u(t_k) - u(t_0)}{t_k - t_0} = \langle (\nabla f)(\vec{b}_k) \, , \, \frac{1}{t_k - t_0}(\vec{x}_k - \vec{a}) \rangle$$

,
then

$$\lim_{k\to\infty} \frac{u(t_k) - u(t_0)}{t_k - t_0} = \langle (\nabla f)(\gamma(t_0)) \, , \, \gamma'(t_0) \rangle$$

□

Proposition (16.4)**.** *$A \neq \emptyset$ in $\mathcal{M}_n$. Let Δ' and Δ'' be two divisions of A. Then exists division Γ of A such that $\Gamma \prec \Delta'$ and $\Gamma \prec \Delta''$.*

Proof. Write $\Delta' = \{A'_1, \cdots, A'_r\}$, $\Delta'' = \{A''_1, \cdots, A''_s\}$.
Put $\Gamma = \{A'_i \cap A''_j \mid 1 \leq i \leq r, 1 \leq j \leq s, A'_i \cup A''_j \neq \emptyset\}$ □

Lemma (16.7)**.** *$A \in \mathcal{M}_n$, $f \colon A \to \mathbb{R}$ bounded. Let Δ, Γ be divisions of A such that $\Gamma \prec \Delta$. Then we have $U(f, \Gamma) \leq U(F, \Delta)$ and $L(f, \Gamma) \geq L(f, \Delta)$.*

Proof. Will prove the inequality for upper sums. Let $\Delta = \{A_1, \cdots, A_r\}$. $\Gamma = \{B_{1,1}, \cdots B_{1,q_1}, \cdots, B_{r,1}, \cdots B_{r,q_r}\}$ where $B_{i,1} \cup \cdots \cup B_{i,q_i} = A_i$, $1 \leq i$
$leqr$.

Then

$$U(f, \Gamma) = \sum_{i=1}^{r} \left(\sum_{j=1}^{q_i} \mathrm{Vol}(B_{i,j}) \cdot \sup_{b_{i,j}}(f) \right) \leq \sum_{i=1}^{r} \left(\sum_{j=1}^{q_i} \mathrm{Vol}(B_{i,j}) \right) \cdot \sup_{A_i}(f) = U(f, \Delta).$$

□

Proposition (17.1)**.** *$A \in \mathcal{M}_n$. $f\colon A \to \mathbb{R}$ bounded. The set of real numbers*

$$T = \{U(f, \Delta) \mid \Delta \text{ division of } A\}$$

is bounded from below, so has an inf.

The number $\inf(T) \in \mathbb{R}$ is called the upper integral of f on Am denoted as $\overline{\int}_A f$ or $\overline{\int}_A f(\vec{x})\mathrm{d}\vec{x}$.

The set of real numbers

$$S = \{L(f, \Delta) \mid \Delta \text{ division of } A\}$$

is bounded from above, so has an sup.

The number $\sup(S) \in \mathbb{R}$ is called the lower integral of f on Am denoted as $\underline{\int}_A f$ or $\underline{\int}_A f(\vec{x})\mathrm{d}\vec{x}$.

One has $\underline{\int}_A f \leq \overline{\int}_A f$

Proof. Fix a division if Δ'' of A then $L(f, \Delta'')$ is a lower bound for $T = \{U(f, \Delta') \mid \Delta' \text{ division of } A\}$. Hence T is bounded below with $\inf(T) \geq L(f, \Delta'')$. □

Theorem (17.3)**.** *$A \in \mathcal{M}_n$, $f\colon A \to \mathbb{R}$ bounded. Then TFAE*

1. *f is integrable on A.*
2. *for every $\epsilon > 0$ there exists a division Δ of A such that $U(f, \Delta) - L(f, \Delta) < \epsilon$.*
3. *There exists a sequence $(\Delta_k)_{k=1}^{\infty}$ of divisions of A such that $U(f, \Delta_k) - L(f, \Delta_k) \to 0$.*

Proof. Will prove (1) $\to$ (2). Others are left as exercises.

Denote $\int_A f = I$. So have $\underline{\int}_A f = I = \overline{\int}_A f$. Given $\epsilon > 0$, we need to find a division Δ of A such that $U(f, \Delta) - L(f, \Delta) < \epsilon$.

The idea is to find Δ' such that $I \leq U(f, \Delta') < I + \epsilon/2$. Find Δ'' such that $I - \epsilon/2 < L(f, \Delta'') \leq I$. Then let $\Delta \prec \Delta'$ and $\Delta \prec \Delta''$. Then we find such Δ. □

Proposition (19.1)**.** *Let A be a non-empty set in $\mathcal{M}_n$, and let f be a function in $Int_b(A, \mathbb{R})$. Let $B \in \mathcal{M}_n$ be such that $B \supseteq A$, and let $g\colon B \to \mathbb{R}$ be defined by*

$$g(\vec{x}) = \begin{cases} f(\vec{x}) & \text{if } \vec{x} \in A \\ 0 & \text{if } \vec{x} \in B \setminus A. \end{cases}$$

Then $f \in Int_b(B, \mathbb{R})$ and $\int_B g = \int_A f$.

Corollary (19.2)**.** *Suppose that $A, B \in \mathcal{M}_n$ such that $A \subseteq B$. Let $I_A\colon B \to \mathbb{R}$ be the indicator function defined by*

$$I_A(\vec{x}) = \begin{cases} 1 & \text{if } \vec{x} \in A \\ 0 & \text{if } \vec{x} \in B \setminus A \end{cases}$$

Then $I_A \in Int_b(B, \mathbb{R})$, and $\int_B I_A = vol(A)$.

Corollary (19.3)**.** *Suppose that $A_1, A_2, \cdots, A_p \in \mathcal{M}_n$ are non-empty sets in $\mathcal{M}_n$ such that $A_i \cap A_j =$ for $i \neq j$. Suppose moreovef that we are given some functions $f_1 \in Int_b(A_1, \mathbb{R}), \cdots f_p \in Int_b(A_p, \mathbb{R})$. Consider the union $A = A_1 \cup \cdots \cup A_p$, and let $f\colon A \to \mathbb{R}$ be defined by*

$$f(\vec{x}) = \begin{cases} f_1(\vec{x}) \text{ if } \vec{x} \in A_1 \\ \cdots \\ f_p(\vec{x}) \text{ if } \vec{x} \in A_p \end{cases}$$

Then $f \in Int_b(A, \mathbb{R})$ and $\int_A f = \int_{A_1} f_1 + \cdots + \int_{A_p} f_p$.