MATH 148 Calculus II, Final Definitions and Theorems

1 Definition (Left Riemann Sum)**.** Given $f(x)$, bounded on $[a,b]$, and a partition $\mathcal{P}$ of $[a,b]$, we define the *left Riemann sum of $f(x)$ with respect to $\mathcal{P}$* by

$$L(f,\mathcal{P}) = L_a^b(f,\mathcal{P}) = \sum_{i=1}^{n} m_i \cdot \Delta x_i,$$

where

$$m_i = \inf\{f(x)\colon x \in [x_{i-1}, x_i]\},$$

and

$$\Delta x_i = x_i - x_{i-1}.$$

2 Definition (Right Riemann Sum)**.** Given $f(x)$, bounded on $[a,b]$, and a partition $\mathcal{P}$ of $[a,b]$, we define the *right Riemann sum of $f(x)$ with respect to $\mathcal{P}$* by

$$U(f,\mathcal{P}) = U_a^b(f,\mathcal{P}) = \sum_{i=1}^{n} M_i \cdot \Delta x_i,$$

where

$$M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\}.$$

3 Definition (Integrability of Functions)**.** We say that $f(x)$ is *Riemann integrable* on $[a,b]$ if

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x,$$

where

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\{U(f,\mathcal{P})\colon \mathcal{P} \text{ a partition of } [a,b]\},$$

and

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\{L(f,\mathcal{P})\colon \mathcal{P} \text{ a partition of } [a,b]\},$$

4 Theorem. *If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of $f(x)$ on $[a,b]$, then $L(f,\mathcal{P}) \leq U(f,\mathcal{Q})$.*

Proof. Let $\mathcal{T} = \mathcal{P} \cup \mathcal{Q}$. Then $\mathcal{T}$ refines both $\mathcal{P}$ and $\mathcal{Q}$. Hence $L(f,\mathcal{P}) \leq L(f,\mathcal{T}) \leq U(f,\mathcal{T}) \leq U(f,\mathcal{Q})$, as desired. □

5 Theorem. *If there exists a sequence of partitions $\mathcal{P}_N$ such that $\lim_{n\to\infty} L(f,\mathcal{P}_n) = \lim_{n\to\infty} U(f,\mathcal{P}_n) = R$ then f is integrable.*

Proof. □

6 Theorem. *If f is bounded on $[a,b]$ then f is integrable on $[a,b]$ if and only if for every $\epsilon > 0$ there is a partition $\mathcal{P}$ such that*

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon$$

Proof. First suppose that for every $\epsilon > 0$ there is a partition satisfying

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon.$$

Since $\inf\{U(f,\mathcal{P}')\colon \mathcal{P}'\} \leq U(f,\mathcal{P})$ and $\sup\{L(f,\mathcal{P}')\colon \mathcal{P}'\} \geq L(f,\mathcal{P})$ it follows that

$$\inf\{U(f,\mathcal{P}')\colon \mathcal{P}'\} - \sup\{L(f,\mathcal{P}')\colon \mathcal{P}'\} \leq U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon$$

for all ϵ. Hence $\inf\{U(f,\mathcal{P}')\colon \mathcal{P}'\} - \sup\{L(f,\mathcal{P}')\colon \mathcal{P}'\}$ must be zero. Hence f is integrable.

Now suppose f is integrable, i.e., $\inf\{U(f,\mathcal{P}')\colon \mathcal{P}'\} = \sup\{L(f,\mathcal{P}')\colon \mathcal{P}'\}$. Hence there are partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$U(f,\mathcal{P}_1) - L(f,\mathcal{P}_2) < \epsilon$$

Let $\mathcal{P}$ be any partition containing $\mathcal{P}_1$ and $\mathcal{P}_2$. Since $U(f,\mathcal{P}) \leq U(f,\mathcal{P}_1)$ and $L(f,\mathcal{P}) \geq L(f,\mathcal{P}_2)$. It follows that for partition $\mathcal{P}$ the equation is satisfied. □

7 Theorem. *Suppose $a < c < b$. If f is integrable on $[a, b]$ then f is integrable on $[a, c]$ and $[c, b]$ and vice versa. If f is integrable on $[a, b]$ then*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Suppose f is integrable on $[a, b]$. Take any $\epsilon > 0$. There is a partition $P = \{a = t_0, t_1, \cdots, t_n = b\}$ of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. If P does not contain c then construct a new partition by adding the point c to it. Hence we can assume that P contains c. That is $t_k = c$ for some k. Then $P_1 = \{a = t_0, t_1, \cdots, t_k = c\}$ and $P_2 = \{c = t_k, t_{k+1}, \cdots, t_n = b\}$ are partitions of $[a, c]$ and $[c, b]$. Since

$$L(f, P) = L(f, P_1) + L(f, P_2)$$

$$U(f, P) = U(f, P_1) + L(f, P_2)$$

we have

$$U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) = U(f, P) - L(f, P) < \epsilon.$$

Since both $U(f, P_1) - L(f, P_1)$ and $U(f, P_2) - L(f, P_2)$ are positive, hence each must be smaller than ϵ. Hence f is integrable on $[a, c]$ and $[c, b]$.

Furthermore, by definition of integrability,

$$L(f, P_1) \leq \int_a^c f \leq U(f, P_1)$$

$$L(f, P_2) \leq \int_b^c f \leq U(f, P_2)$$

so

$$L(f, P) = L(f, P_1) + L(f, P_2) \leq \int_a^c f + \int_c^b f \leq U(f, P_1) + U(f, P_2) \leq U(f, P)$$

proving that

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Now we show that if f is integrable on $[a, c]$ and $[c, b]$ then it is integrable on $[a, b]$. Let P_1 and P_2 be partitions of $[a, b]$ and $[c, b]$ such that $U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}$ and $U(f, P_2) - L(f, P_2) < \frac{\epsilon}{2}$. Let P be the partition containing the points of P_1 and P_2. Then

$$U(f, P) - L(f, P) = U(f, P_1) + U(f, P_2) - L(f, P_1) - L(f, P_2) < \epsilon$$

proving that f is integrable on $[a, b]$. $\square$

8 Theorem. *If f and g are integrable on $[a, b]$ then $f + g$ is integrable on $[a, b]$ and*

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g$$

Proof. Let $P = [t_0, \cdots, t_N]$ be a partition of $[a, b]$. Let $Q_k = [t_{k-1}, t_k]$ for $k = 1, 2, \cdots, N$. Since

$$\inf(f, Q_k) + \inf(g, Q_k) \leq \inf(f + g, Q_k)$$

then

$$L(f, P) + L(g, P) \leq L(f + g, P).$$

Similarly,

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

Hence

$$L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P).$$

Since f and g are integrable there are partitions P_f and P_g for which $U(f, P_f) - L(f, P_f) < \frac{\epsilon}{2}$ and $U(g, P_g) - L(g, P_g) < \frac{\epsilon}{2}$. Let P contain both P_f and P_g, then

$$U(f, P) + U(g, P) - L(f, P) - L(g, P) < \epsilon$$

and it follows that

$$U(f + g, P) - L(f + g, P) < \epsilon.$$

Hence $f + g$ is integrable. $\square$

9 Theorem. *If f is integrable on $[a,b]$ and F is defined on $[a,b]$ by*

$$F(x) = \int_a^x f,$$

then F is continuous on $[a,b]$.

Proof. By definition f is bounded on $[a,b]$ so there is a number M such that $|f| \leq M$ on $[a,b]$.

We prove continuity at a point inside $[a,b]$. The proof for $x = a$ or $x = b$ is similar. Take $x \in (a,b)$ and choose any $\epsilon > 0$. Let $\delta = \min\{x-a, b-x, \frac{\epsilon}{M}\}$. Then if $|x-y| < \delta$

1. $y \in [a,b]$
2. $F(y)$ is defined
3. $M|x-y| < \epsilon$

Then if $|x-y| < \delta$ we have

$$F(x) - F(y) = \int_y^x f$$

and since

$$m \leq f(x) \leq M \text{ on } [a,b] \Rightarrow m(b-a) \leq \int_a^b f \leq M(b-a)$$

hence

$$-M|x-y| \leq \int_y^x f \leq M|x-y|$$

or

$$|F(x) - F(y)| < \epsilon.$$

Hence F is continuous at x. □

10 Theorem (The First Fundamental Theorem of Calculus)**.** *Let f be integrable on $[a,b]$ and define $F(x)$ by*

$$F(x) = \int_a^x f.$$

If f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$. If $c = a$ or b then $F'(c)$ means the right- or -left derivative of F.

Proof. Since f is continuous at c, then for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x| < \delta$ then $|f(c+x) - f(c)| < \epsilon$.

Then if $0 < h < \delta$, then

$$F(c+h) - F(c) = \int_c^{c+h} f$$

lies between $h(f(c)+\epsilon)$ and $h(f(c)-\epsilon)$ since f is between $f(c)-\epsilon$ and $f(c)+\epsilon$ on $[c, c+h]$.

Then

$$\frac{F(c+h) - F(c)}{h} \in (f(c)-\epsilon, f(c)+\epsilon)$$

so

$$\lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c).$$

□

11 Theorem (The Second Fundamental Theorem of Calculus)**.** *If f is integrable on $[a,b]$ and $f = g'$ for some function g then*

$$\int_a^b f = g(b) - g(a)$$

Proof. Let $P = \{t_0, \cdots, t_n\}$ be any partition of $[a,b]$. By the Mean Value Theorem there is a point $x_i \in [t_{i-1}, t_i]$ such that $g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1})$.

On each sub-interval we have

$$m_i \leq f(x_i) \leq M_i$$

hence

$$L(f,P) \leq \sum_i f(x_i)(t_i - t_{i-1}) \leq U(f,P)$$

But

$$\sum_i f(x_i)(t_i - t_{i-1}) = \sum_i g(t_i) - g(t_{i-1}) = g(b) - g(a)$$

Thus

$$L(f, P) \leq g(b) - g(a) \leq U(f, P)$$

for all partitions P. This means that $g(b) - g(a) = \int_a^b f$. $\square$

12 Definition (Taylor Polynomial)**.** The Taylor Polynomial of Degree n for f at a is

$$P_{n,a}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$$

where

$$a_k = \frac{f^{(k)}(a)}{k!}$$

Suppose $f^{(n+1)}$ is continuous on $[a,x]$. Then

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-n)^n \, \mathrm{d}t$$

is called the integral form of the remainder.

13 Definition (Summability)**.** The sequence $\{a_n\}$ is summable if the sequence s_n where

$$s_n = \sum_{k=1}^{n} a_k$$

converges to some number s as $n \to \infty$. The s_n are called partial sums.

14 Definition (Cauchy Criterion)**.** The Cauchy Criterion says that the sequence $\{a_n\}$ is summable iff

$$\lim_{m,n\to\infty} a_{n+1} + \cdots + a_m = 0.$$

In other words, it is summable iff

$$\lim_{m,n\to\infty} (s_m - s_n) \to 0$$

15 Definition (Absolutely Convergence)**.** A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges. A series that converges but does not converge absolutely is said to converge conditionally.

16 Definition (Cauchy Product)**.** The Cauchy Product of two series is a particular ordering of the $a_i b_j$

$$\sum_{n=0}^{\infty}(\sum_{i=0}^{n} a_i b_{n-i})$$

17 Definition (Uniform Convergence)**.** Let $\{f_n\}$ be a sequence of functions defined on a set A and let $f(x)$ be a function defined on A such that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in A$.

If for every $\epsilon > 0$ there is an N such that for all $n > N$ we have

$$|f_n(x) - f(x)| < \epsilon \text{ for all } x \in A$$

then the f_n are said to converge uniformly to $f(x)$ on A.

18 Definition (Series of Functions)**.** The series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly to $f(x)$ on A if the sequence of functions

$$s_n(x) = \sum_{k=0}^{n} f_k(x)$$

converges uniformly to f on A.

19 Theorem. *If the sequence* $\{a_n\}$ *is summable then*

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Use the Cauchy Criterion with $m = n+1$. Since

$$\lim_{n\to\infty} s_n = s$$

then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

□

20 Theorem (Comparison Test)**.** *Suppose $0 \le a_n \le b_n$ for all $n > M$. Then if $\sum b_n$ converges so does $\sum a_n$.*

Proof. Let

$$M_a = \sum_{k=1}^{M} a_k$$

$$M_b = \sum_{k=1}^{M} b_k$$

and let

$$s_n = \sum_{k=1}^{n} a_k = M_a + \sum_{k=M+1}^{n} a_k$$

$$t_n = \sum_{k=1}^{n} b_k = M_b + \sum_{k=M+1}^{n} b_k$$

for $n > M$.

Then

$$0 \le s_n - M_a \le t_n - M_b$$

for all $n > M$ and these are increasing sequences for $n > M$. Because $\sum b_k$ exists, then the sequence $t_n - M_b$ is bounded. Hence $s_n - M_a$ is bounded. Hence $\sum a_k$ exists. □

21 Theorem (Limit Comparison Test)**.** *Suppose that $\{a_n\}$ and $\{b_n\}$ are non-negative sequences such that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Then

1. *If $L < \infty$ and $\sum b_n$ converges then $\sum a_n$ converges.*
2. *If $L > 0$ and $\sum b_n$ diverges then so does $\sum a_n$*

22 Theorem (Ratio Test)**.** *Let $a_n > 0$ for all $n > M$ and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$. Then*

1. *$\sum a_n$ converges if $r < 1$.*
2. *if $r > 1$ then $a_n \to \infty$ as $n \to \infty$ and the series diverges.*

Proof. When $r < 1$, choose any s with $r < s < 1$. Then there is an $N > M$ such that $0 < \frac{a_{n+1}}{a_n} < s$ for $n > N$.

Since $\sum s^k a_N = a_n \sum s^k$ converges so does $\sum a_{N+k}$. Hence so does

$$\sum a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n = \sum_{n=0}^{n} a_n + \sum a_{N+k}$$

When $r > 1$ choose any s with $1 < s < r$. There is an $N > M$ such that for all $n > N$ we have $a_{n+1} > sa_n$. This means that $a_{N+k} > s^k a_N$ for all $k > 0$. Hence $a_n \to \infty$ as $n \to \infty$ since $a_n > 0$ and $s > 1$. □

23 Theorem (Integral Test)**.** *Suppose $f(x)$ is positive and decreasing on $[1, \infty)$. Let $a_n = f(n)$ for all positive integers n. Then $\sum a_n$ converges iff*

$$\int_1^\infty f$$

exists.

Proof. Let P_N be the partition $\{1, 2, \cdots, N\}$. The sum $\sum_{n=1}^{N-1} a_n$ is the left Riemann sum $R_{L,N}$ for f on P_N, since f is decreasing

Hence if $\int_1^\infty f$ exists so does $\lim_{N\to\infty} R_{R,N}$ because the sequence $R_{R,N}$ is increasing and bounded by $\int_1^\infty f$, hence $\sum_{n=1}^{\infty} a_n$ exists. □

24 Theorem (Every absolute convergent series is convergent)**.**

1. *Every absolute convergent series is convergent.*
2. *A series is absolutely convergent iff the series formed from its positive terms and the series formed from its negative terms both converge.*

Proof. Let

$$a_n^+ = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases}$$

$$a_n^- = \begin{cases} a_n & \text{if } a_n \leq 0 \\ 0 & \text{if } a_n > 0 \end{cases}$$

Note that

$$a_n = a_n^+ + a_n^-$$

$$|a_n| = a_n^+ - a_n^-$$

Hence the increasing sequence $\sum a_n^+$ is bounded by $\sum |a_n|$, so does the increasing sequence $\sum a_n^-$. □

25 Theorem (Cauchy criterion for uniform convergence)**.** *$f_n(x)$ converge uniformly to $f(x)$ on A if and only if for every $\epsilon > 0$ there exists an integer n_0 such that $m \geq n_0$ and $n \geq n_0$ implies that*

$$|f_m(x) - f_n(x)| < \epsilon$$

for every $x \in A$.

Proof. Suppose that

$$f_n(x) \to f \text{ uniformly on } A.$$

Choose $\epsilon > 0$. Then there is an n_0 such that for $n > n_0$ we have $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in A$. Then for all $m \geq n_0$ and $n \geq n_0$ we have

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

Now suppose that for any $\epsilon > 0$ there is an n_0 such that for all $m > n_0$ and $n > n_0$ we have

$$|f_n(x) - f_m(x)| < \epsilon$$

for all $x \in A$. By the Cauchy Criterion for convergence we know that $f_n(x)$ has a limit $f(x)$ as $n \to \infty$ for each x.

Choose $\epsilon > 0$. Choose n_0 such that $n \geq n_0$ implies

$$|f_n(x) - f_{n+k}(x)| < \epsilon/2$$

for all $x \in A$ and for all $k \geq 1$. Then

$$\lim_{k\to\infty} |f_n(x) - f_{n+k}(x)| = |f_n(x) - f(x)| \leq \epsilon/2$$

Hence for all $n \geq n_0$ we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Hence f_n converge uniformly on A. □

26 Theorem (Uniform Convergence and Integration)**.** *Suppose that $\{f_n\}$ is a sequence of functions that are integrable on $[a, b]$ and that they converge uniformly on $[a, b]$ to a function f which is also integrable on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \lim_n \to \infty f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

27 Theorem (Uniform Convergence and Continuity)**.** *Suppose that $\{f_n\}$ is a sequence of functions that are continuous on $[a, b]$ that they converge uniformly on $[a, b]$ to a function f. Then f is continuous on $[a, b]$.*

Proof. Consider $x \in (a, b)$. We need to show that $\lim_{h\to 0} f(x + h) = f(x)$. Since $f_n \to f$ uniformly on $[a, b]$ then there is an n such that

$$|f(y) - f_n(y)| < \epsilon/3 \text{ for all } y \in [a, b]$$

So for all h for which $x + h$ is in $[a, b]$ we have

$$|f(x) - f_n(x)| < \epsilon/3$$

$$|f(x + h) - f_n(x + h)| < \epsilon/3$$

Because f_n is continuous there is a $\delta > 0$ such that for $|h| < \delta$ we have

$$|f_n(x+h) - f_n(x)| < \epsilon/3$$

Hence if $|h| < \delta$ we have

$$\begin{aligned}|f(x+h) - f(x)| &= |f(x+h) - f_n(x+h) + f_n(x+h) - f_n(x) + f_n(x) - f(x)| \\ &\leq |f(x+h) - f_n(x+h)| + |f_n(x+h) - f_n(x)| + |f_n(x) - f(x)| \\ &< \epsilon\end{aligned}$$

which proves that $f(x)$ is continuous. □

28 Theorem (Uniform Convergence and Differentiation)**.** *Let $\{f_n\}$ be a sequence of functions differentiable on a closed finite interval $[a, b]$ with integrable derivatives f_n' and suppose that f_n converge pointwise to f. Suppose also that f_n' converge uniformly on $[a, b]$ to a continous function g. Then f is differentiable and*

$$f' = \lim_{n\to\infty} f_n'(x)$$

29 Theorem (Weierstrass M-test)**.** *Let $\{f_n\}$ be a sequence of functions defined on A and suppose that M_n is a sequence of numbers such that*

$$|f_n(x)| \leq M_n$$

for all $x \in A$. Then if $\sum M_n$ converges then $\sum f_n(x)$ converges absolutely on A and converges uniformly to $f(x) = \sum f_n(x)$.