MATH 146 Linear Algebra, Final Theorem

1 Required Theorems

1 Theorem (2.42)**.** *For $A \in M_{n\times n}(\mathbb{F})$, the following are equivalent:*

1. *A is invertible*
2. $\operatorname{rank}(A) = n$
3. *A can be written as a product of elementary matrices.*
4. *A can be transformed by elementary row operations to I_n.*

Proof. (1) $\Leftrightarrow$ (2): A is invertible $\iff$ L_A is a bijection $\iff$ L_A is surjective $\iff$ $R(L_A) = \mathbb{F}^n \iff \dim(R(L_A)) = n \iff \operatorname{rank}(A) = n$.

(2) $\Rightarrow$ (3): If $\operatorname{rank}(A) = n$, then A can be transformed by elementary operations to I_n and

$$\begin{aligned} I_n &= E_k \cdots E_2 E_1 A E_1' E_2' \cdots E_l' \\ &= PAQ. \end{aligned}$$

Since P, Q are invertible we get

$$\begin{aligned} A &= P^{-1}(PAQ)Q^{-1} \\ &= P^{-1} I_n Q^{-1} \\ &= P^{-1}Q^{-1} \\ &= (E_1)^{-1}(E_2)^{-1} \cdots (E_k)^{-1}(E_l')^{-1} \cdots (E_2')^{-1}(E_1')^{-1}. \end{aligned}$$

As the inverse of an elementary matrix is again an elementary matrix, this proves (3).

(3) $\Rightarrow$ (4): Assume that $A = E_1 E_2 \cdots E_k$ where each E_i is elementary. Then A is invertible, each E_i^{-1} is also elementary, and $A_{-1} = E_k{-1} \cdots E_2^{-1} E_1^{-1}$. Thus

$$I_n = A^{-1}A = E_k{-1} \cdots E_2^{-1} E_1^{-1} A.$$

Multiplying on the left by elementary matrices is the same as applying elementary row operations. Thus this equations shows that A can be transformed by elementary row operations to I_n.

(4) $\Rightarrow$ (2): $\operatorname{rank}(I_n) = n$ and elementary operations preserve rank. □

2 Theorem (4.16)**.** *For all $A, B \in M_{n\times n}(\mathbb{F})$,* $\det(AB) = \det(A)\det(B)$.

Proof. Case 1: $\operatorname{rank}(A) < n$. Then $\operatorname{rank}(AB) \le \operatorname{rank}(A) < n$. Thus $\det(A) = \det(AB) = 0$.

Case 2: $\operatorname{rank}(A) = n$. Then A is invertible, so can be written as a product of elementary matrices

$$A = E_1 E_2 \cdots E_k.$$

Thus

$$\begin{aligned} \det(AB) &= \det(E_1 E_2 \cdots E_k B) \\ &= \det(E_1)\det(E_2)\cdots\det(E_k)\det(B) \\ &= \det(E_1 E_2 \cdots E_k)\det(B) \\ &= \det(A)\det(B). \end{aligned}$$

□

3 Corollary (4.18)**.** $\det(A^t) = \det(A)$.

Proof. We already know this for elementary matrices. Now consider cases.

Case 1: $\operatorname{rank}(A) < n$. Then $\operatorname{rank}(A^t) < n$ as well so $\det(A) = 0 = \det(A^t)$.

Case 2: $\operatorname{rank}(A) = n$. Then we can write $A = E_1 E_2 \cdots E_k$ with each E_i elementary. Then

$$\begin{aligned} \det(A^t) &= \det((E_k)^t)\cdots\det((E_2)^t)\det((E_1)^t) \\ &= \det(E_k)\cdots\det(E_2)\det(E_1) \\ &= \det(A). \end{aligned}$$

□

4 Theorem (5.7)**.** *Suppose V is finite-dimensional, $T \in L(V)$, λ is an eigenvalue of T, and m is the multiplicity of λ. Then* $\dim(E_\lambda) \leq m$*.*

Proof. Let $d = \dim(E_\lambda)$. Let $\alpha = (v_1, \cdots, v_d)$ be an ordered basis for E_λ. Extend α to an ordered basis $\beta = (v_1 \cdots, v_d, v_{d+1}, \cdots, v_n)$ for V. Let $A = [T]_\beta$, so $p_T(t) = p_A(t)$.

Observe that for $i = 1, \cdots, d$,

$$\begin{aligned} T(v_i) &= \lambda v_i \\ &= 0v_1 + \cdots \lambda v_i + \cdots + 0v_n \end{aligned}$$

Hence

$$A = \left(\begin{array}{cccc|ccc} \lambda & 0 & \cdots & 0 & * & \cdots & * \\ 0 & \lambda & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & * & \cdots & * \\ 0 & 0 & \cdots & \lambda & * & \cdots & * \\ \hline 0 & 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & * & \cdots & * \end{array}\right) = \begin{pmatrix} \lambda I_d & B \\ O & C \end{pmatrix}$$

for some matrices of the appropriate dimensions. Thus

$$\begin{aligned} p_T(t) = \det(A - tI_n) = \begin{pmatrix} (\lambda - t)I_d & B \\ O & C - tI_{n-d} \end{pmatrix} \\ &= \det((\lambda - t)I_d) \det(C - tI_{n-d}) \\ &= (\lambda - t)^d \det(I_d) \det(c - tI_{n-d}) \\ &= (\lambda - t)^d \cdot p_C(t). \end{aligned}$$

This proves that $(t - \lambda)^d$ divides $p_T(t)$. Since m, the multiplicity of λ, is by definition the largest value such that $(t - \lambda)^m$ divides $p_T(t)$, we have proved that $d \leq m$, i.e., $\dim(E_\lambda) < m$. ☐

2 Theorems which I think is important

5 Theorem (5.3)**.** *Let $A \in M_{n\times n}(\mathbb{F})$.*

1. *$p_A(t)$ is a polynomial in $\mathbb{F}[t]$ of degree n.*
2. *The leading coefficient of $p_A(t)$ is $(-1)^n$.*
3. *The coefficient of t^{-1} is $(-1)^{n-1}\operatorname{tr}(A)$*

6 Theorem (5.1)**.** *Suppose V is a finite-dimensional vector space over $\mathbb{F}$ and $T \in L(V)$. T is diagonalizable iff V has an ordered basis consisting of eigenvectors of T.*

3 Unfamiliar Definition

7 Definition (Eigenvector)**.** Let $A \in M_{n\times n}(\mathbb{F})$. An eigenvector of A is any non-zero vector $v \in \mathbb{F}$ satisfying $Av \in \text{span}(v)$. The unique scalar $\lambda \in \mathbb{F}$ satisfying $Av = \lambda v$ is the eigenvalue of A corresponding to v.

8 Definition (Eigenvector II)**.** Let V be a finite-dimensional vector space over $\mathbb{F}$ and let $T \in L(V)$. An eigenvector of T is any nonzero vector $v \in V$ satisfying $T(v) \in \text{span}(v)$. The unique scalar $\lambda \in \mathbb{F}$ satisfying $T(v) = \lambda v$ is the eigenvalue of T corresponding to v.

9 Definition (Eigenspace)**.** Suppose $T \in L(V)$.

1. Given an eigenvalue λ of T, the set

$$\begin{aligned} E_\lambda &= \{v \in V : T(v) = \lambda v\} \\ &= \{\text{eigenvectors of } T \text{ corresponding to } \lambda\} \cup \{0\}. \end{aligned}$$

2. The spectrum of T is the set of eigenvalues of T.

10 Definition (Characteristic polynomial)**.** Let $A \in M_{n\times n}(\mathbb{F})$. The characteristic polynomial of A is the expression $\det(A - tI_n)$. It is denoted $p_A(t)$.

11 Definition (Similar)**.** Suppose $A, B \in M_{n\times n}(\mathbb{F})$. We say that B is similar to A over F if there exists an invertible $Q \in M_{n\times n}(\mathbb{F})$ such that $B = Q^{-1}AQ$.

12 Definition (Split)**.** A polynomial $f(t) \in \mathbb{F}[t]$ splits over $\mathbb{F}$ if there exist scalars $c, a_1, \cdots a_n \in \mathbb{F}$ not necessarily distinct such that

$$f(t) = c(t - a_1)\cdots(t - a_n)$$

13 Definition (Multiplicity)**.** Suppose $T \in L(V)$ and λ is an eigenvalue of T. The multiplicity of λ is the maximum value k such that $(t - \lambda)^k$ is a factor of $p_T(t)$.

14 Definition (Diagonalizable)**.** Suppose V is finite-dimensional and $T \in L(V)$. T is diagonalizable if there exists an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Suppose $A \in M_{n\times n}(\mathbb{F})$. A is diagonalizable over $\mathbb{F}$ if A is similar over $\mathbb{F}$ to a diagonal matrix D.