MATH 147 LECTURE NOTES

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My Lecture Notes for MATH 147 2016 Fall

December 2016

Lecture 1, Sept. 12

Mathematical tools $\LaTeX$

MikTex, Winshell

Basics on Sets and Functions

1.1 Definition. Basic Sets

- $\mathbb{N}$ = Natural numbers = $\{1, 2, 3, \ldots\}$
- $\mathbb{Z}$ = Integers = $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{Q} = \{\frac{m}{n} \mid n \in \mathbb{N}, m \in \mathbb{Z}, gcd(n, |m|) = 1\}$
- $\mathbb{R}$ = Real Numbers
- $\mathbb{R}\backslash\mathbb{Q} = \{x \in \mathbb{R} \mid x \text{ is not in } \mathbb{Q}\}$

Notation.

$S \subset X \rightarrow S$ is a subset of X

If $S, T \subset X$ then $S \cup T = \{x \in X \mid x \in S \text{ or } x \in T\}$

If $S, T \subset X$ then $S \cap T = \{x \in X \mid x \in S \text{ and } x \in T\}$

Given a collection $\{A_\alpha\}_{\alpha \in I}$ of subsets of X

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for some } \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for all } \alpha \in I\}$$

$\emptyset$ = empty set, $\emptyset \subset X$

What if $I = \emptyset$, what is $\bigcup_{\alpha \in \emptyset} A_\alpha$

Define

$$\bigcup_{\alpha \in \emptyset} A_\alpha = \emptyset$$

Then

$$\bigcap_{\alpha \in \emptyset} A_\alpha = ??$$

Given $S, T \subset X$ we define

$$S\backslash T = \{x \in X \mid x \in S, x \text{ does not belong to } T\}$$

We denote $X\backslash T$ by T^c = compliment of T in $X = \{x \in X \mid x \text{ does not belong to } T\}$

Note.

$$(S \cup T)^c = S^c \cap T^c$$

De Morgans Law

1.2 Theorem.

$$(\bigcup_{\alpha \in I} A_\alpha)^c = \bigcap_{\alpha \in I} A_\alpha^c$$

Proof.

$$\begin{aligned} x \in (\bigcup_{\alpha \in I} A_\alpha)^c &\iff x \text{ is not a member of } \bigcup_{\alpha \in I} A_\alpha \\ &\iff x \text{ is not in } A_\alpha \quad \forall \alpha \in I \\ &\iff x \in A_\alpha^c \quad \forall \alpha \in I \\ &\iff x \in \bigcap_{\alpha \in I} A_\alpha^c \end{aligned}$$

□

Note. From this we really should have

$$\begin{aligned} \bigcap_{\alpha \in \emptyset} A_\alpha &= (\bigcup_{\alpha \in \emptyset} A_\alpha^c)^c \\ &= \emptyset^c \\ &= X \end{aligned}$$

Power Set

1.3 Definition. Given X, the Power Set of X is the set of all subset of X

Notation.

$$\begin{aligned} P(X) &= \text{ power set of } X \\ &= \{S \mid S \subset X\} \end{aligned}$$

Note. We can observe that

$$\emptyset, X \in P(X)$$

Lecture 2, Sept. 14

New Section 12:30-1:20 CPH 3604
Tutorial Moved to DC 1350 Th 4:30-5:20

Greek Letters

- α - alpha
- β - beta
- δ - delta
- ϵ - epsilon
- γ - gamma

Properties of $\mathbb{N}$

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

Mathematical Induction

2.1 Axiom. Assume $S \in \mathbb{N}$ such that

1. $1 \in S$
2. If $k \in S$, then $k + 1 \in S$

Then $S = \mathbb{N}$

Proof by Induction

1. Establish for each $n \in \mathbb{N}$ a statement $P(n)$ to be proved.
   Example. Let $P(n)$ be the statement that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, show this is true for all $n \in \mathbb{N}$.
   Let $S = \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$, show $S = \mathbb{N}$
2. Base Case: show that $P(1)$ is true. ie): $1 \in S$
3. Inductive Step: Assume that $P(k)$ is true for some k (Inductive Hypothesis). Use this to show that $P(k + 1)$ is also true. ie): $k \in S \Rightarrow k + 1 \in S$

By the Principle of Mathematical Induction, $S = \mathbb{N}$

2.2 Example. Prove that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof. Step.1 Let $P(n)$ be the statement that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Step.2 Let $n = 1$ then $P(1) = 1 = \frac{1(1+1}{2}$. Hence $P(1)$ is true.

Step.3 Assume that $P(k)$ is rue for some k

$$P(k)\frac{k(k+1}{2}$$

Step.4

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

Hence $P(k+1)$ is true

Step.5 By Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$ □

2.3 Example. Prove that $3^n + 4^n$ is divisible by 7 for every odd n

Proof. Let $P(k)$ be the statement that $3^{2k-1} + 4^{2k-1}$ is divisible by 7.

Base case: $k = 1$, $P(1)$ is true.

Inductive Step: Assume $P(j)$ is true.

$$\begin{aligned}&3^{2(j+1)-1} + 4^{2(j+1)-1} \\ =&9(3^{2j-1}) + 16(4^{2j-1}) \\ =&9(3^{2j-1} + 4^{2j-1}) + 7(4^{2j-1})\end{aligned}$$

Hence $P(j+1)$ is true.

By Principle of Mathematical Induction, $P(k)$ is true for all n

□

Lecture 3, Sept. 16

Well Ordering Property

3.1 Theorem. *If $S \in \mathbb{N}$ and $S \neq \emptyset$, then S contains a least element.*

The following are equivalent

1. Principle of Mathematical Induction
2. Strong Induction
3. Well Ordering Principle

Note. A function f such that $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ can be defined by $f((m,n)) = 7^n 13^m$

Properties of $\mathbb{R}$

Interval

3.2 Theorem. *A set $I \in \mathbb{R}$ is an interval if for each $x, y \in I$ with $x \leq y$ and $z \in I$ with $x \leq y \leq z$, we have $z \in I$*

3.3 Question. 1. Is $\emptyset$ an interval? Yes

2. Is $\{3\}$ an interval? Yes

Other Intervals

1. $[a, b] = x \in \mathbb{R} \mid a \leq x \leq b \to$ Closed Interval
2. $(a, b) = x \in \mathbb{R} \mid a < x < b \to$ Open Interval
3. $[a, b) = x \in \mathbb{R} \mid a \leq x < b \to$ Half Open Half Closed Interval
4. $[a, \infty) = x \in \mathbb{R} \mid a \leq x \to$ Closed Ray
5. $(\infty, b] = x \in \mathbb{R} \mid x \leq b \to$ Closed Ray
6. $(0, \infty)$
7. $(-\infty, b)$
8. $(-\infty, \infty) = \mathbb{R}$

Lecture 4, Sept. 19

Least Upper Bound Property

Upper Bound

4.1 Theorem. *Let $S \subset \mathbb{R}$ then $\alpha \in \mathbb{R}$ is an upper bound for S if $x \leq \alpha$ for all $x \in S$. We say that S is bounded above if S has an upper bound.*

We say that β is a lower bound for S if $\beta \leq x$ for all $x \in S$. We say that S is bounded below if S has a lower bound.

We say that S is bounded if it is bounded above and below.

4.2 Example. Let $S = \{x_1, x_2, \ldots, x_n\}$ be finite.

By relabeling, if necessary we can assume that

$$x_1 < x_2 < \cdots < x_n$$

Then $\beta = x_1$, β is a lower bound and $\alpha = x_n$ is an upper bound.

4.3 Theorem. *Every finite set is bounded.*

4.4 Example. Let $S = [0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ (finite interval)

5 is an upper bound. -1 is a lower bound.

1 is also an upper bound. Moreover if γ is any upper bound of S, then $1 \leq \gamma$

Least Upper Bound

4.5 Theorem. *We say that α is the least upper bound of a set $S \subset \mathbb{R}$ if*

1) *α is an uppper bound of S*
2) *if γ is an upper bound of S, then $\alpha \leq \gamma$*

We write

$$\alpha = lub(S)$$

(Sometimes α os called the supremum of S and is denoted by $\alpha = sup(S)$)

Back to the example $S = [0, 1)$. 0 is a lower bound and if γ is any lower bound, then $\gamma \leq 0$

Greatest Upper Bound

4.6 Theorem. *We say that β is the greatest lower bound of a set $S \subset \mathbb{R}$ if*

1) *β is an lower bound of S*
2) *if γ is an lower bound of S, then $\gamma \leq \beta$*

We write

$$\beta = glb(S)$$

(Sometimes β os called the infimum of S and is denoted by $\beta = inf(S)$)

4.7 Example. if $S = [0, 1)$, $lub(S) = 1$, $glb(S) = 0$.

Note. Is $\emptyset$ bounded (above or below)?

Note: 6 is an upper bound for $\emptyset$. If not, there exists an element in $\emptyset$ that is greater than 6. Similarly, 6 is a lower bound.

In fact, if $\gamma \in \mathbb{R}$ then γ is both an upper and a lower bound of $\emptyset$. $\emptyset$ is a bounded set.

4.8 Example. Let $S = \{x \in \mathbb{Q} \mid x^2 < 2\} \subset \mathbb{R}$

$\sqrt{2}$ is an upper bound and $-\sqrt{2}$ is a lower bound. And $lub(S) = \sqrt{2}$, $glb(S) = -\sqrt{2}$

4.9 Example. Let $S = \{x \in \mathbb{Q} \mid x^2 < 2\} \subset \mathbb{Q}$

S does not have a least upper bound or a greatest lower bound.

4.10 Question. If $S \subset R$ is bounded above, does it always have a least upper bound?

Least Upper Bound Property

4.11 Theorem. *If $S \subset R$ is non-empty and bounded above, then S has a least upper bound.*

Observation

1) $\emptyset$ does not have a *lub*
2) If we only have rational numbers in the world, then $S = \{x \mid x^2 < 2\}$ does not have a lub. In other words, Least Upper Bound Property fails for $\mathbb{Q}$

4.12 Question. is $\mathbb{N}$ bounded?

1) $\mathbb{N}$ is bounded below, $glb(S) = 1$

Lecture 5, Sept. 21

1) No office hours this afternoon

2) WA1→ Due 2:30 PM Monday, Sept. 26. Submit in dropbox outside Math Tutorial Center.

Least Upper Bound Property If $S \subset \mathbb{R}$ is non-empty and bounded above, then S has a least upper bound.

Archimedean Property I

5.1 Theorem. $\mathbb{N}$ *is not bounded above.*

Proof. Suppose that $\mathbb{N}$ was bounded above. Then $\mathbb{N}$ has a least upper bound α.

Note that $\alpha - \frac{1}{2} < \alpha$. Hence $\alpha - \frac{1}{2}$ is not an upper bound for $\mathbb{N}$. Then there exists $n \in \mathbb{N}$ with $\alpha - \frac{1}{2} < n \leq \alpha$. But then $n+1 \in \mathbb{N}$ and $n+1 > \alpha$ which is impossible.

Therefore $\mathbb{N}$ must not be bounded above.

□

Note. Let $S \neq \emptyset \subset \mathbb{R}$ be bounded above. Let $\alpha = lub(S)$. if $\epsilon > 0$ then there exist $x_0 \in S$ with $\alpha - \epsilon < x_0 \leq \alpha$.

Archimedean Property II

5.2 Corollary. *Let* $\epsilon > 0$*, Then there exists* $n \in \mathbb{N}$ *such that*

$$0 < \frac{1}{n} < \epsilon$$

Proof. Take $\alpha = \frac{1}{\epsilon}$ in Archimedean Property I. □

Density of $\mathbb{R}$

5.3 Definition. A subset $S \subset \mathbb{R}$ is said to be dense if for every $\epsilon > 0$ and $x \in \mathbb{R}$,

$$S \cap (x - \epsilon, x + \epsilon) \neq \emptyset$$

or equivalently if $S \cap (a, b) \neq \emptyset$ for all $a < b$ in $\mathbb{R}$

5.4 Proposition. $\mathbb{Q}$ *and* $\mathbb{R} \backslash \mathbb{Q} = \mathbb{Q}^c$ *are dense in* $\mathbb{R}$

Absolute Values

5.5 Definition.

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

5.6 Example.

$$g(x) = \frac{|x|}{x}$$

Domain $= \{x \in \mathbb{R} \mid x \neq 0\}$

$$g(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Geometric Interpretation of $|x|$

- $|x|$ represents the distance from x to 0.
- $|x - a|$ represents the distance from x to a.

Note. Distance between $(0, 0)$ and (x, y)

$$\sqrt{x^2 + y^2}$$

Properties of $|x|$

1) $|x| \geq 0$ and $|x| = 0 \iff x = 0$
2) $|ax| = |a||x|$ for all $a \in \mathbb{R}, x \in \mathbb{R}$
3) Triangle Inequality

$$|x - z| + |z - y| \geq |x - y|$$

5.7 Theorem. ***Triangle Inequality*** *If $x, y, z \in \mathbb{R}$, then*

$$|x - z| + |z - y| \geq |x - y|$$

Proof. Use Geometric Interpretation. □

5.8 Theorem. ***Variants I*** *For all $x, y \in \mathbb{R}$,*

$$|x + y| \leq |x| + |y|$$

5.9 Theorem. ***Variants II*** *For all $x, y \in \mathbb{R}$,*

$$||x| - |y|| \leq |x - y|$$

Lecture 6, Sept. 23

Inequalities

6.1 Example. Find all $x \in \mathbb{R}$ such that

$$0 < |x - 2| \leq 4$$

Solution. $[-2, 6]$ with $x \neq 2$

Three Basic Inequalities

1. $|x - a| < \delta$
2. $0 < |x - a| < \delta$
3. $|x - a| \leq \delta$

Solution. 1. $(a - \delta, a + \delta) = \{x \in R \mid a - \delta < x < a + \delta\}$

2. $(a - \delta, a + \delta)$ with $x \neq a = \{x \in \mathbb{R} \mid a - \delta < x < a + \delta, x \neq a\}$
3. $[a - \delta, a + \delta] = \{x \in R \mid a - \delta \leq x \leq a + \delta\}$

Sequence

6.2 Definition. A **sequence** is an infinite ordered list of real numbers.

Notation. $\{1, 2, 3, 4, \dots\}$ or $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

6.3 Definition. A **sequence** of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$

The element $f(n)$ is called the n-th term of the sequence. We often denote this by $f_(n) = a_n$

Notation. We can denote sequences in many ways

1. $f(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$
2. Let $a_n = \frac{1}{n}$
3. $\{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$
4. $\{\frac{1}{n}\}$
5. Sometimes we define sequences recursively.
   $a_1 = 1$ and $a_{n+1} = \sqrt{3 + 2a_n}$ for all $n \geq 1$.

Graphing Sequence

Subsequence

6.4 Definition. Let $\{a_n\}$ be a sequence, and let $\{n_k\}$ be a sequence of natural numbers with $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \dots$.

The sequence $b_k = a_{n_k} \to \{b_k\}_{k=1}^{\infty}$ is called **subsequence** of $\{a_n\}$. We often write this as

$$\{a_{n_1}, a_{n_2}, \dots, a_{n_k}, \dots\}$$

Important Subsequences Given $\{a_n\}$, let $n_0 \in \mathbb{N} \cup \{0\}$. Define

$$b_k = a_{n_0+k}$$

This sequence is called a tail of $\{a_n\}$

Limits of Sequences Consider $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$

Note. As n gets larger and larger, the terms of the sequence $\{\frac{1}{n}\}$ get closer and closer to 0. We would like to say that the sequence $\{\frac{1}{n}\}$ converges to 0 and call 0 the limit of $\{\frac{1}{n}\}$.

6.5 Definition. (Heuristic Definition of Convergence). We say that a sequence $\{a_n\}$ has a limit L if for every positive tolerance $\epsilon > 0$, the term a_n will approximate L with an error less than ϵ so long as the index n is large enough.

Lecture 7, Sept. 26

Writing Assignment 2 is due Friday Oct 14th.

Convergence of Sequences

7.1 Definition. Heuristic definition I We say that a sequence $\{a_n\}$ converges to a limit L if as n gets larger and larger the a_ns get closer and closer to L.

7.2 Definition. Heuristic definition II We say that a sequence $\{a_n\}$ converges to a limit L if for every positive tolerance $\epsilon > 0$, we have that the terms in $\{a_n\}$ approximate L with an error at most ϵ, provided that n is large enough.

7.3 Definition. Convergence of a Sequence We say that $\{a_n\}$ converges to a limit L if for every $\epsilon > 0$, there exists a cutoff $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then $|a_n - L| < \epsilon$

If no such L exists, we say that $\{a_n\}$ **diverges**.

7.4 Example. Consider $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \dots\}$. Does this have a limit?

Proof. Let $\epsilon = 1$. Suppose $L = \lim_{n\to\infty} a_n$. Let N_0 be such that if $n \geq N_0$, then $|a - L| < 1$

Let $n_1 \geq N_0$ with n_0 even. Then

$$\begin{aligned} |-1-L| &= |a_n - L| < 1 \\ &\to L \in (-2, 0) \end{aligned}$$

Let $n_1 \geq N_0$ with n_0 odd. Then

$$\begin{aligned} |1-L| &= |a_n - L| < 1 \\ &\to L \in (0, 2) \end{aligned}$$

So

$$L \in (-2, 0) \cap (0, 2)$$

which is impossible.

Hence $\{a_n\}$ diverges. □

Note. Suppose that $\lim_{n\to\infty} a_n = L$. Let $\epsilon > 0$. What can we say about the terms in $\{a_n\}$ that are in $(L - \epsilon, L + \epsilon)$?

For some N_0, if $n \geq N_0$, then $a_n \in (L - \epsilon, L + \epsilon)$. ie) $(L - \epsilon, L + \epsilon)$ contains a tail of the sequence.

7.5 Proposition. *Let $\{a_n\}$ be a sequence. Then the following are equivalent.*

1. $L = \lim_{n\to\infty} a_n$
2. *for every* $\epsilon > 0$, $(L - \epsilon, L + \epsilon)$ *contains a tail of* $\{a_n\}$
3. *for every* $\epsilon > 0$, $(L - \epsilon, L + \epsilon)$ *contains all but finitely many* a_n
4. *for open interval* (a, b) *with* $L \in (a, b)$, *we have* (a, b) *contains a tail of* $\{a_n\}$

5. *for open interval* (a, b) *with* $L \in (a, b)$*, the interval* (a, b) *contains all but finitely many* a_n

7.6 Question. Can $\{a_n\}$ have more than 1 limit?

7.7 Theorem. ***Uniqueness of Limit*** *Suppose that* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} a_n = M$*, then* $L = M$

Proof. Assume that $L < M$. Let $\epsilon = \frac{M-L}{2}$.

We can choose N_1 large enough so that if $n \geq N_1$, $a_n \in (L - \epsilon, L + \epsilon)$

We can also choose N_2 large enough so that if $n \geq N_2$, $a_n \in (M - \epsilon, M + \epsilon)$

Let $N_0 = max\{N_1, N_2\}$. Choose $n \geq N_0$. Then $a_n \in (L - \epsilon, L + \epsilon) \cap (M - \epsilon, M + \epsilon)$

But $(L - \epsilon, L + \epsilon) \cap (M - \epsilon, M + \epsilon) = \emptyset$

□

Lecture 8, Sept. 28

8.1 Theorem. *Assume that $\{a_n\}$ converges. then $\{a_n\}$ is bounded.*

Proof. Assume that

$$L = \lim_{n\to\infty} a_n$$

Let $\epsilon = 1$. Then there exists $N_0 \in \mathbb{N}$ so that if $n \geq N_0$ then $|a_n - L| < 1$

If $n \geq N_0$,then

$$\begin{aligned} |a_n| = |a_n - L + L| &\leq |a_n - l| + |L| \\ &< 1 + |L| \end{aligned}$$

Let

$$M = max|a_1|, |a_2|, \ldots, |a_{N_0-1}|, |L| + 1$$

.

Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. ☐

Question: Do all bounded sequences converge? No.

8.2 Definition.

1. We say that a sequence $\{a_n\}$ is increasing if $a_n < a_{\{}n+1\}$ for all $n \in \mathbb{N}$
2. We say that $\{a_n\}$ is non-decreasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$
3. We say that $\{a_n\}$ is decreasing if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$
4. We say that $\{a_n\}$ is non-increasing if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$

We say that $\{a_n\}$ is monotonic if $\{a_n\}$ satisfies one of the conditions.

Example:

1. $$\{a_n\} = \{\frac{1}{n}\}$$
   is decreasing, since
   $$\frac{1}{n+1} \leq \frac{1}{n}$$
   for all $n \in \mathbb{N}$
2. $$\{cos(n)\}$$
3. Let $a_1 = 1$,
   $$a_{n+1} = \sqrt{3 + 2a_n}$$

8.3 Theorem. ***Monotone Convergence Theorem***

If $\{a_n\}$ is monotonic and bounded, then $\{a_n\}$ converges.

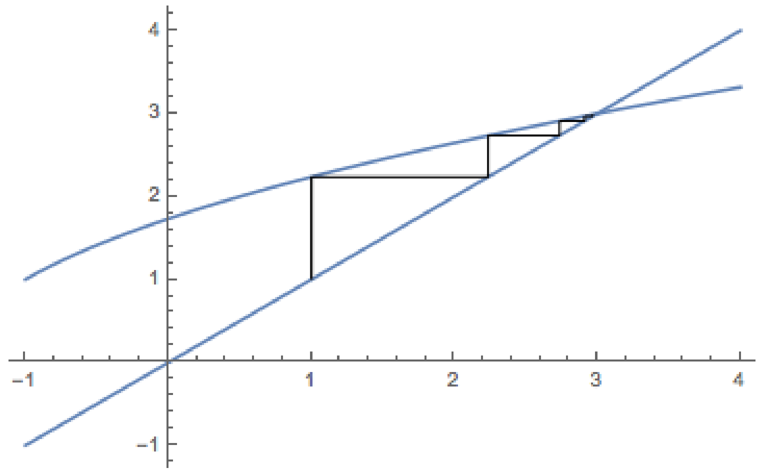


Figure 1: $y = \sqrt{3+2x}$ and $y = x$

Proof. Assume that $\{a_n\}$ is non-decreasing and bounded above. Let $L = lub(\{a_n\})$

Let $\epsilon > 0$, then $L - \epsilon$ is not an upper bound. Then there exists $N_0 \in \mathbb{N}$ so that $L - \epsilon < a_{N_0} \leq L$. If $n \geq N_0$, then $L - \epsilon < a_{N_0} \leq a_n \leq L$, so $|a_n - L| < \epsilon$. Hence $L = \lim_{n\to\infty} a_n$

Similarly, if $\{a_n\}$ is non-increasing then $L = \lim_{n\to\infty} a_n$ where $L = glb(\{a_n\})$ □

8.4 Example. Let $a_1 = 1$,

$$a_{n+1} = \sqrt{3 + 2a_n}$$

We know that $0 \leq a_n < a_{n+1} \leq 3$ for all $n \in \mathbb{N}$. $\{a_n\}$ is increasing and bounded above. Hence $\{a_n\}$ converges.

8.5 Corollary. *A monotonic sequence $\{a_n\}$ converges iff it is bounded.*

8.6 Definition. We say a sequence **diverges to** ∞ if for every $M > 0$ we can find a a cutoff $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then $M \leq a_n$, we write $\lim_{n\to\infty} a_n = \infty$.

Lecture 9, Sept. 29

9.1 Definition. We sat that $\{a_n\}$ diverges to ∞ if for every $M \geq 0$ there exists $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then $a_n > M$

We write

$$\lim_{n\to\infty} a_n = \infty$$

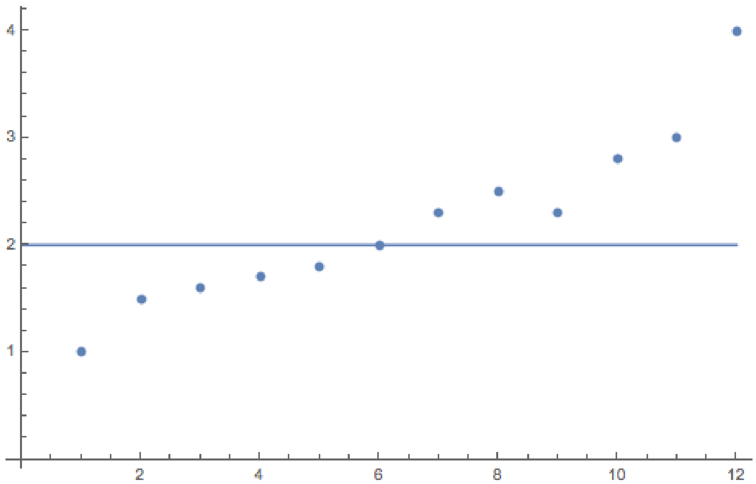


Figure 2: $\{a_n\}$ and $M = 2$

9.2 Question. Does every sequence $\{a_n\}$ that is <u>not</u> bounded above diverges to ∞?

No. $\{0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \dots\}$

Note. If $\{a_n\}$ is non-decreasing then either

1) $\{a_n\}$ is bounded and convergent
2) $\{a_n\}$ is unbounded and diverges to ∞

9.3 Question. If a sequence is not bounded above, does it have a sub-sequence that diverges to ∞?

Series

Given a Sequence $\{a_n\}$, what does it mean to sum all of the terms of the sequence? That is what does the formal sum mean

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

9.4 Example.

$$\sum_{n=1}^{\infty} (-1)^{n+1}$$

9.5 Example.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

9.6 Definition. For each $k \in \mathbb{N}$, the kth partial sum is

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

We say that $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\{S_k\}$ of partial sums converges. Otherwise we say the series diverges.

If the series converges we let

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n$$

9.7 Example.

$$\sum_{n=1}^{\infty} (-1)^{n+1}$$

$$S_k = \begin{cases} 1 & \textit{if } k \textit{ is odd} \\ 0 & \textit{if } k \textit{ is even} \end{cases}$$

Thus S_k diverges

Geometric Series Let $r \in R$, consider

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots$$

$$S_k = \sum_{n=0}^{k} r^n = 1 + r + r^2 + r^3 + \cdots + r^k$$

$$S_k = \frac{1 - r^{k+1}}{1 - r} \quad \textit{if } r \neq 1$$

Note. If $|r| < 1$ then $\lim_{k\to\infty} r^{k+1} = 0$

If $|r| > 1$ then $\lim_{k\to\infty} r^{k+1}$ does not exists

If $r = -1$ then $\lim_{k\to\infty} r^{k+1}$ does not exists.

If $r = 1$ then $S_k = k$ which diverges to infinity.

9.8 Example. $r = \frac{1}{2}$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - \frac{1}{2}} = 2$$

Lecture 10, Sept. 30

Series

10.1 Definition. A series $\sum_{n=1}^{\infty} a_n$ is **positive** is for all $n \in \mathbb{N}$, if $S_k = \sum_{n=1}^{k} a_n$, then $S_{k+1} - S_k = a_{k+1} \geq 0$

10.2 Example. Harmonic Series Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

Let $S_k = \displaystyle\sum_{n=1}^{k} \frac{1}{n}$,

$$\begin{aligned}
S_1 &= 1 = \frac{2}{2} \\
S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{4}{2} \\
S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{2} \\
&\vdots \\
S_{2^k} &> \frac{2+k}{2}
\end{aligned}$$

Since $\{\frac{2+k}{2}\}$ is not bounded, $\{S_k\}$ is not bounded.

10.3 Example. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$

Note.

$$\begin{aligned}
\frac{1}{n^2 - n} &= \frac{1}{n(n-1)} \\
&= \frac{1}{n-1} - \frac{1}{n}
\end{aligned}$$

Solution.

$$\begin{aligned}
S_1 &= 1 - \frac{1}{2} = 1 - \frac{1}{2} \\
S_2 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3} \\
S_3 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4} \\
&\vdots \\
S_k &= 1 - \frac{1}{k}
\end{aligned}$$

As $k \to \infty$, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - n} = 1$

10.4 Example. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Note. For $n \geq 2$,

$$\frac{1}{n^2} < \frac{1}{n^2 - n}$$

$$\begin{aligned} T_k = \sum_{n=1}^{k} \frac{1}{n^2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \\ &< 1 + \frac{1}{2^2 - 2} + \frac{1}{3^2 - 2} + \cdots + \frac{1}{k^2 - k} \\ &< 1 + 1 \\ &= 2 \end{aligned}$$

Since $T_k \leq 2$ for all k, $\{T_k\}$ is bounded and by the Monotone Convergence Theorem is convergent with $1 \leq \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$.

In fact, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

10.5 Example. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$, does this converge?

Note that $\dfrac{1}{n!} < \dfrac{1}{2^n}$ for $n \geq k$.

In fact, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} = e$

Note.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Arithmetic Rules for Sequences

10.6 Question. Assume $a_n \to 3$, $b_n \to 7$.

What can you say about

1) $\{4a_n\}$
2) $\{a_n b_n\}$
3) $\{a_n + b_n\}$

4) $\{\frac{a_n}{b_n}\}$

10.7 Theorem. ***Arithmetic Rules for Sequences*** *Let* $\{a_n\}$, $\{b_n\}$ *be such that* $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$.

Then

1) $\lim_{n\to\infty} ca_n = cL$ *for all* $c \in \mathbb{R}$

2) $\lim_{n\to\infty} a_n + b_n = L + M$

3) $\lim_{n\to\infty} a_n b_n = LM$

4) $\lim\limits_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$ *if* $L \neq 0$

5) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ *if* $M \neq 0$

Proof. 1) If $c = 0$ then $ca_n = 0$ for all n. Hence $\lim_{n\to\infty} ca_n = \lim_{n\to\infty} 0 = 0L = cL$ Suppose $c \neq 0$, Let $\epsilon > 0$. We want N so that if $n \geq N$, $|ca_n - cL| < \epsilon \Leftrightarrow |a_n - L| < \frac{\epsilon}{|c|}$

Choose N_0 such that if $n \geq N_0$ we have $|a_n - L| < \frac{\epsilon}{|c|}$

If $n \geq N_0$,

$$|ca_n - cL| \leq |a_n - L||c| < \frac{\epsilon}{|c|}|c| = \epsilon$$

□

Lecture 11, Oct. 3

WA2 now due Monday Oct. 17

EA2 due today

11.1 Theorem. ***Arithmetic Rules for Sequences*** *Let* $\{a_n\}$, $\{b_n\}$ *be such that* $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$.

Then

1) $\lim_{n\to\infty} ca_n = cL$ *for all* $c \in \mathbb{R}$
2) $\lim_{n\to\infty} a_n + b_n = L + M$
3) $\lim_{n\to\infty} a_n b_n = LM$
4) $\lim\limits_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$ *if* $L \neq 0$
5) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ *if* $M \neq 0$
6) $\lim\limits_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{L}$ *if* $L \geq 0$

Proof.

1) If $c = 0$ then $ca_n = 0$ for all n. Hence $\lim_{n\to\infty} ca_n = \lim_{n\to\infty} 0 = 0L = cL$ Suppose $c \neq 0$, Let $\epsilon > 0$. We want N so that if $n \geq N$, $|ca_n - cL| < \epsilon \Leftrightarrow |a_n - L| < \frac{\epsilon}{|c|}$

   Choose N_0 such that if $n \geq N_0$ we have $|a_n - L| < \frac{\epsilon}{|c|}$

   If $n \geq N_0$,
$$|ca_n - cL| \leq |a_n - L||c| < \frac{\epsilon}{|c|}|c| = \epsilon$$

2) Consider
$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |a_n - L + b_n - M| \\ &\leq |a_n - L| + |b_n - M| \end{aligned}$$

   Let $\epsilon > 0$. Choose $N_1 \in \mathbb{N}$ so that
$$n \geq N_1 \to |a_n - L| < \frac{\epsilon}{2}$$

   Choose $N_2 \in \mathbb{N}$ so that
$$n \geq N_2 \to |b_n - M| < \frac{\epsilon}{2}$$

   Let $N_0 = max\{N_1, N_2\}$. If $n \geq N_0$
$$\begin{aligned} |(a_n + b_n) - (L + M)| &\leq |a_n - L| + |b_n - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

3) Consider $|a_n b_n - LM|$

$$\begin{aligned} &|a_n b_n - LM| \\ =&|a_n b_n - b_n L + b_n L - LM| \\ =&|(a_n - L)b_n + L(b_n - M)| \\ \leq&|(a_n - L)b_n| + |L(b_n - M)| \end{aligned}$$

By 1), we can find N_1 so that if $n \geq N_1$,

$$|L|\,|b_n - M| \leq \frac{\epsilon}{2}$$

Since $\{b_n\}$ is convergent it is bounded. So there exists $c > 0$ so that $|b_n| < c$
Then $|b_n|\,|a_n - L| < c\,|a_n - L|$
Choose N_2 so that if $n \geq N_2$

$$|a_n - L| < \frac{\epsilon}{2c}$$

If $N_0 = max\{N_1, N_2\}$ and $n \geq N_0$ then

$$|a_n b_n - LM| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

4)

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|a_n - L|}{|a_n|\,|L|}$$

Since $a_n \to L, L \neq 0$ we can find $N_1 \in \mathbb{N}$ so that if $n \geq N_1$, then

$$|a_n - L| < \frac{|L|}{2} \to |a_n| \geq \frac{|L|}{2}$$

If $n \geq N_1$ then

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|a_n - L|}{|a_n|\,|L|} \leq \frac{|a_n - L|}{\frac{|L|}{2}|L|} = \frac{|a_n - L|}{\frac{|L|^2}{2}}$$

Let $\epsilon > 0$. Choose N_2 so that if $n \geq N_2$

$$\frac{|a_n - L|}{\frac{|L|^2}{2}} < \epsilon$$

Let $N_0 = max\{N_1, N_2\}$ if $n \geq N_0$

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| < \epsilon$$

5) Follows from 3 and 4.

6) Homework

□

Note. If $\lim_{n\to\infty} a_n = L, \lim_{n\to\infty} n_n = M$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} \text{ if } M \neq 0$$

What happens if $M = 0$?

It depends on a_n.

11.2 Example. $a_n = b_n = \frac{1}{n}$

11.3 Example. $a_n = \frac{1}{n}, b_n = \frac{1}{n^2}$

11.4 Proposition. *Assume that* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and that* $\lim_{n\to\infty} b_n = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Proof.

$$\begin{aligned} a_n &= (b_n)(\frac{a_n}{b_n}) \\ &= \lim_{n\to\infty} a_n \\ &= \lim_{n\to\infty} b_n \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= 0L \\ &= 0 \end{aligned}$$

□

Lecture 12, Oct. 5

12.1 Example. Find $\dfrac{3n^2+2n}{5n^2+2}$

Solution.

$$\begin{aligned}
\lim_{n\to\infty}\frac{3n^2+2n}{5n^2+2} &= \lim_{n\to\infty}\frac{n^2}{n^2}\frac{3+\dfrac{2}{n}}{5+\dfrac{2}{n^2}} \\
&= \frac{\lim\limits_{n\to\infty}3+\lim\limits_{n\to\infty}\dfrac{2}{n}}{\lim\limits_{n\to\infty}5+\lim\limits_{n\to\infty}\dfrac{2}{n^2}} \\
&= \frac{3+0}{5+0} \\
&= \frac{3}{5}
\end{aligned}$$

Note. If $a_k b_j \neq 0$

$$\lim_{n\to\infty}\frac{a_0+a_1n+\cdots+a_kn^k}{b_0+b_1n+\cdots+b_jn^j} = \begin{cases} \dfrac{a_k}{b_j} & \textit{if } k=j \\ 0 & \textit{if } j>k \\ \infty & \textit{if } j<k, a_kb_j>0 \\ -\infty & \textit{if } j<k, a_kb_j<0 \end{cases}$$

12.2 Example. Find

$$\lim_{n\to\infty}\sqrt{n^2+n}-n$$

Solution.

$$\begin{aligned}
\lim_{n\to\infty}\sqrt{n^2+n}-n &= \lim_{n\to\infty}(\sqrt{n^2+n}-n)\cdot\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty}\frac{n}{\sqrt{1+\dfrac{1}{n}}+1} \\
&= \frac{n}{\sqrt{1+\lim\limits_{n\to\infty}\dfrac{1}{n}}+1} \\
&= \frac{1}{2}
\end{aligned}$$

12.3 Example. $a_1 = 1$ and $a_{n+1} = \dfrac{1}{1+a_n}$. Suppose that $\{a_n\}$ converges, find $\lim_{n\to\infty} a_n$

12.4 Proposition. *A sequence $\{a_n\}$ converges to L if and only if every sub-sequence $\{a_{n_k}\}$ converges to L*

Proof. Assume that $lim_{n\to\infty} a_n = L$. Let $\{a_{n_k}\}$ be a sub-sequence. Let $\epsilon > 0$, we can find a N_0 so that if $n \geq N_0$, then $|a_n - L| < \epsilon$.

Let $k_0 \geq N_0$, then $k \geq k_0 \Rightarrow n_k \geq n_{k_0} \geq N_0$

Hence $|a_{n_k} - L| < \epsilon$ ☐

Solution. If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$

Then,

$$L = \frac{1}{1+L}$$
$$L^2 + L - 1 = 0$$
$$L = \frac{-1 \pm \sqrt{5}}{2}$$

12.5 Question. Does $\{a_n\}$ converge?

Solution. Claim that for any k,

$$a_{2k} < a_{2k+2} < a_{2k+1} < a_{2k-1}$$

Proof by induction.

$\{a_{2k-1}\}$ is decreasing and bounded below by 0

$\{a_{2k}\}$ is increasing and bounded above by 1.

Let $\lim_{n\to\infty} a_{2k} = M$ and $\lim_{n\to\infty} a_{2k-1} = L$.

Since $M = \dfrac{-1+\sqrt{5}}{2}$ and $L = \dfrac{-1+\sqrt{5}}{2}$, $M = L$

Thus, $\{a_n\}$ converges.

12.6 Example. FInd

$$\lim_{n\to\infty} \frac{cos(n)}{n}$$

Lecture 13, Oct. 6

Squeeze Theorem

13.1 Example. Find

$$\lim_{n\to\infty} \frac{cos(n)}{n}$$

Observation:

$$|cos(n)| \leq 1$$

$$\frac{-1}{n} \leq \frac{cos(n)}{n} \leq \frac{1}{n}$$

13.2 Theorem. ***Squeeze Theorem*** *If* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *are such that* $a_n \leq b_n \leq c_n$ *with* $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, *then* $\lim_{n\to\infty} b_n = L$

Proof. Let $\epsilon > 0$, then exists $N_0 \in \mathbb{N}$ so that if $n \geq N_0$ then $a_n \in (L-\epsilon, L+\epsilon)$ and $c_n \in (L-\epsilon, L+\epsilon)$

If $n \geq N_0$,

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

$$|b_n - L| < \epsilon$$

□

Solution. We know that

$$\frac{-1}{n} \leq \frac{cos(n)}{n} \leq \frac{1}{n}$$

since $|cos(n)| \leq 1$

Since $\lim_{n\to\infty} -\frac{1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$

Then

$$\lim_{n\to\infty} \frac{cos(n)}{n} = 0$$

13.3 Example.

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$$

Note. If $\{a_n\}$ is bounded, then

$$\lim_{n\to\infty} \frac{a_n}{n} = 0$$

Bolzano-Weierstrass Theorem

Note. We know that convergent sequences are bounded. But bounded sequences do not have to converge.

Does every bounded sequences have a convergent sub-sequence?

Strategy Bounded + monotonic ⇒ convergent

Does every sequence have a monotonic sub-sequence

13.4 Definition. Given $\{a_n\}$ we call an index n_0 a **peak point** for $\{a_n\}$ if $a_n < a_{n_0}$ for all $n \geq n_0$

13.5 Lemma. ***Peak Point Lemma*** *Every sequence $\{a_n\}$ has a monotonic sub-sequence.*

Proof. Let $P = \{n \in \mathbb{N} \mid n \textit{ is a peak point of } \{a_n\}\}$

Case 1. P is infinite.

Let $n_1 =$ least element of P

Let $n_2 =$ least element of P
$\{n_1\}$

...

This gives us a sequence recursively

$$n_1 < n_2 < \cdots < n_k < \cdots \in P$$

Since these are peak points,

$$a_{n_k} > a_{n_{k+1}}$$

Thus $\{a_{n_k}\}$ is decreasing.

Case 2. Let n_1 be the least index that is not a peak point. Since n_1 is not a peak point, we can choose $n_2 > n_1$ so that

$$a_{n_1} \leq a_{n_2}$$

Since n_2 is not a peak point, then we can choose $n_3 > n_2$ so that

$$a_{n_2} \leq a_{n_3}$$

We can proceed recursively, to find that

$$n_1 < n_2 < \cdots < n_k < \ldots$$

Where $a_{n_k} \leq a_{n_{k+1}}$

Thus $\{a_{n_k}\}$ is non-decreasing.

In either case we have a monotonic sub-sequence. □

13.6 Theorem. ***Bolzano-Weierstrass Theorem*** *Every bounded sequences has a convergent sub-sequence.*

Proof. Give $\{a_n\}$, by the Peak Point Lemma $\{a_n\}$ has a monotinic subsequence $\{a_{n_k}\}$, which is also bounded. By the MCT, $\{a_{n_k}\}$ is convergent. □

Note. BWT is equivalent to MCT which is equivalent to the LUBP.

Lecture 14, Oct. 7

14.1 Theorem. ***Bolzano-Weierstrass Theorem*** *Every bounded sequences has a convergent sub-sequence.*

14.2 Definition. We say that $\alpha \in \mathbb{R}$ is a **limit point** of $\{a_n\}$ if there exists a sub-sequence $\{a_{n_k}\}$ with $\lim_{n\to\infty} a_{n_k} = \alpha$

LET $LIM(\{a_n\}) = \{\alpha \in \mathbb{R} \mid \alpha$ is a limit point of $\{a_n\}$

14.3 Example. $a_n = (-1)^{n+1} \to \{1, -1, 1, -1, \dots\}$

$LIM(\{a_n\}) = \{1, -1\}$

14.4 Example. $a_n = n \to \{1, 2, 3, \dots\}$

$LIM(\{a_n\}) = \emptyset$

Fact If $\{a_n\}$ converges with $\lim_{n\to\infty} a_n = L$, then $LIM(\{a_n\}) = \{L\}$

14.5 Question. If $\{a_n\}$ is such that $LIM(\{a_n\})$ contains only one value α, does $\{a_n\}$ converges to α?

No. Counterexample:

$$\{a_n\} = \{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \dots\}$$

14.6 Proposition. *α is a limit point of $\{a_n\}$ if for every $(\alpha - \epsilon, \alpha + \epsilon)$ contains infinite many terms of the sequence.*

Assume α is a limit point of $\{a_n\}$, then there exists a sub-squence $\{a_{n_k}\}$ with $a_{n_k} \to \alpha$. There exists $K_0 \in \mathbb{N}$ so that $k \geq K_0 \to |a_{n_k} - \alpha| < \epsilon \to a_{n_k} \in (\alpha - \epsilon, \alpha + \epsilon)$

Proof. Assume that $\forall \epsilon > 0$, $(\alpha - \epsilon, \alpha + \epsilon)$ contains infinitely many terms of $\{a_b\}$

For $\epsilon = 1$ we can find n_1 so that $a_{n_1} \in (\alpha - 1, \alpha + 1)$

$a_{n_2} \in (\alpha - \frac{1}{2}, \alpha + \frac{1}{2})$

Suppose we have $n_1 < n_2 < n_3 < \cdots < n_k$ with

$$a_{n_j} \in (\alpha - \frac{1}{j}, \alpha + \frac{1}{j})$$

Since $(\alpha - \frac{1}{k+1}, \alpha + \frac{1}{k+1})$ contains infinitely many $a_n s$. there is $n_{k+1} > n_k$ with $a_{n_{k+1}} \in (\alpha - \frac{1}{k+1}, \alpha + \frac{1}{k+1})$

We proceed recursively to get a sub-sequence $\{a_{n_k}\}$ with

$$a_{n_k} = (\alpha - \frac{1}{k}, \alpha + \frac{1}{k})$$

$$\alpha - \frac{1}{k} < a_{n_k} < \alpha + \frac{1}{k}$$

By the squeeze theorem, $a_{n_k} \to \alpha$

□

14.7 Question.

1. Suppose $\{a_n\}$ is bounded and $LIM(\{a_n\}) = \{L\}$, does $\lim_{n\to\infty} L$?
2. Does there exists $\{a_n\}$ with $LIM(\{a_n\}) = \{R\}$
3. For which subsets S of R does there exists $\{a_n\}$ with $LIM(\{a_n\}) = S$?

Cauchy Sequence

14.8 Question. Is there an intrinsic way to characterize a convergent sequence?

Note. If $\lim_{n\to\infty} a_n = L$ and if $\epsilon > 0$ then we can find N_0 so that if $n \geq N_0$m

$$|a_n - L| < \frac{\epsilon}{2}$$

If $n, m \geq N_0$, then

$$\begin{aligned} |a_n - a_m| &= |(a_n - L) + (L - a_m)| \\ &\leq |a_n - L| + |L - a_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

14.9 Definition. A sequence $\{a_n\}$ is **Cauchy** is for every $\epsilon > 0$, then there exists $N_0 \in \mathbb{N}$ so that if $n, m \geq N_0$, then

$$|a_n - a_m| < \epsilon$$

14.10 Proposition. *Every convergent sequence is Cauchy*

14.11 Question. Does every Cauchy sequence Converges?

14.12 Lemma. *Every Cauchy Sequence is bounded.*

Proof. Let $\epsilon = 1$ and choose N_0 so that if $n, m \geq N_0$, then $|a_n - a_m| < \epsilon$

Hence, if $n \geq N_0$ then

$$|a_n - a_{N_0}| < 1 \to |a_n| \leq |a_{N_0}| + 1$$

□

Let $M = max\{|a_1|, |a_q|, \ldots, |a_{N_0-1}|, |a_{N_0}| + 1\}$

14.13 Lemma. *Let $\{a_n\}$ be Cauchy. Assume that $\{a_{n_k}\}$ is such that $\lim_{k\to\infty} a_{n_k} = L$, then*

$$\lim_{n\to\infty} a_n = L$$

Proof. Let $\epsilon > 0$. We can find a N_0 so that if $n, m \geq N_0$, then

$$|a_n - a_m| < \frac{\epsilon}{2}$$

Let $n \geq N_0$

$$\begin{aligned} |a_n - L| &= |(a_n - a_{n_k}) + (a_{n_k} - L)| \\ &\leq |a_n - a_{n_k}| + |a_{n_k} - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□

14.14 Theorem. ***Completeness Property for*** $\mathbb{R}$ *Every Cauchy Sequence Converges.*

Proof. If a_n is Cauchy, then a_n is bounded. By BWT, a_n has a convergent sub-sequence $\{a_{n_k}\}$. Hence a_n converges. (by Lemma 2.) □

Lecture 15, Oct. 17

Women in Pure Math/Math Finance

Lunch/Workshop

Tuesday, Oct.25

12:30-1:20

MC 5417

Limits of Functions

15.1 Example.

$$f(x) = \frac{x^2 - 1}{x - 1}$$

$domain(f) = \{x \in \mathbb{R} \mid x \neq 1\}$.

Note.

$$f(x) = \frac{(x+1)(x-1)}{x-1} = (x+1) \text{ if } x \neq 1.$$

What can we say about the values of $f(x)$ as x approaches 1? As x gets closer and closer to 1, $f(x)$ gets closer and closer to 2. We want to say that 2 is the limit of $f(x)$ as x approaches 1.

15.2 Definition. Heuristic Definition of Limit I If $f(x)$ is defined on an open interval around $x = a$, except possibly at $x = a$, then we say that L is the limit of $f(x)$ as x approaches a if as x gets closer and closer to a, f(x) gets closer and closer to L

15.3 Definition. Heuristic Definition of Limit II We say that L is the limit of $f(x)$ as x approaches a, if for every positive tolerance $\epsilon > 0$, $f(x)$ approximates L with an error less than ϵ provided that x is close enough to a, and not equal to a.

15.4 Definition. Formal Definition for a Limit of a Function We say that L is the limit of $f(x)$ as x approaches a, if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then

$$|f(x) - L| < \epsilon.$$

In this case we write

$$\lim_{x \to a} f(x) = L.$$

15.5 Example. Show that

$$\lim_{x \to 2} 3x + 1 = 7.$$

Solution. Let $\epsilon > 0$

$$|3x + 1 - 7| = |3x - 6| = 3|x - 2|.$$

We want $|3x + 1 - 7| < \epsilon$. We can make this happen if $|x - 2| < \epsilon/3$

Hence if $\delta = \epsilon/3$, then

$$0 < |x - 2| < \delta = \epsilon/3 \Rightarrow |3x + 1 - 7| = 3|x - 2| < 3\epsilon/3 = \epsilon$$

15.6 Example. $f(x) = mx + b, m \neq 0$

$$\lim_{x \to a} f(x) = ma + b$$

Solution. Given $\epsilon > 0$, chooose $\delta = \epsilon / |m|$

15.7 Example. Show that

$$\lim_{x \to 3} x^2 = 9$$

Solution.

$$\left|x^2 - 9\right| = |x+3|\,|x-3|$$

Let $\epsilon > 0$. We can assume $\delta < 1$.

If $0 < |x-3| < 1 \Rightarrow x \in (2,4)$.

Hence $|x+3| < 7$.

Hence for any $\delta < 1$,

$$0 < |x-3| < 1 \Rightarrow \left|x^2 - 9\right| < 7\,|x-3|\,.$$

Let $\delta = min\{1, \epsilon/7\}$

If $0 < |x-3| < \delta \Rightarrow \left|x^2 - 9\right| \leq 7\,|x-3| = \epsilon$

15.8 Example. Show that

$$\lim_{x \to 1} x^7 + 4x^5 - 3x + 2 = 1$$

Solution. Don't want to do this by $\epsilon - \delta$.

15.9 Example.

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

What is

$$\lim_{x \to 0} f(x)$$

Solution. $\lim_{x \to 0} f(x)$ does not existst.

Assume $\lim_{x \to 0} f(x) = L$. Let $\epsilon = 1/2$. Suppose that $\delta > 0$ is such that $0 < |x - 0| < \delta \Rightarrow |f(x) - L| < \epsilon = 1/2$

Let $x = \delta/2$. Then $L \in (1/2, 3/2)$. Let $x = -\delta/2$. Then $L \in (-3/2, -1/2)$.

$$L \in (1/2, 3/2) \cap (-3/2, -1/2) = \emptyset$$

15.10 Theorem. *If* $\lim_{x \to a} f(x) = L$ *and* $\lim_{x \to a} f(x) = M$*, then* $L = M$*.*

15.11 Theorem. $\lim_{x \to a} f(x) = L$ *if and only if whenever* $\{x_n\}$ *is a sequence with* $x_n \to a$; $x_n \neq a$ *we have that* $f(x_n) \to L$

Lecture 16, Oct. 19

Thursday $\to$ Lecture

Friday $\to$ Tutorial

Basic Fact about Limits For $\lim_{n\to a} f(x)$ to exist $f(x)$ must be defined in some open interval I containing $x = a$, except possibly at $x = a$.

Sequential Characterization of Limits

16.1 Theorem. *Let $f(x)$ be defined in an open interval I containing a, except possibly at $x = a$. Then the following are equivalent.*

1. $$\lim_{x\to a} f(x) = L$$

2. *Whenever $\{x_n\}$ is such that $x_n \to a$ ($x_n \neq a$) we have $f(x) \to L$*

Proof. Assume that $\lim_{x\to a} f(x) = L$. Let $\{x_n\}$ be such that $x_n \to a$, $x_n \neq a$. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. Since $x_n \to a$, we can find a $N_0 \in \mathbb{N}$ so that if $n \geq N_0$, then $0 < |x_n - a| < \delta \Rightarrow |f(x_n) - L| < \epsilon$

Conversely, (prove by contrapositive) assume that L is not the limit. Then there exists $e_0 > 0$ such that for any $\delta > 0$, there exists $x_\delta \in (a - \delta, a + \delta)$, $x_\delta \neq a$ and $|f(x_\delta) - L| \geq e_0$. In particular, for each $n \in \mathbb{N}$, there exists $x_n \in (a - \frac{1}{n}, a + \frac{1}{n})$, $x_n \neq a$, such that $|f(x) - L| \geq e_0$. Hence $x_n \to a$, $x_n \neq a$, but $\{f(x_n)\}$ does not converge to L. □

16.2 Theorem. ***Arithmetic Rules for Limits*** *Assume that* $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$ *then*

1. $lim_{x\to a}(cf)(x) = cL$
2. $lim_{x\to a}(f + g)(x) = L + M$
3. $lim_{x\to a}(fg)(x) = L \cdot M$
4. $lim_{x\to a}(f/g)(x) = L/M$ *if* $M \neq 0$

16.3 Theorem. ***Squeeze Theorem for Limits*** *Assume that on some open interval I containing $x = a$ that*

$$f(x) \leq g(x) \leq h(x)$$

for all $x \in I$, except posibly at $x = a$. If $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ *then*

$$\lim_{x\to a} g(x) = L$$

Remark. 1. Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots$ then

$$\lim_{x\to a} p(x) = p(a)$$

2. Let $f(x) = p(x)/q(x)$ where $p(x), q(x)$ are polynomials, then

$$\lim_{x\to a} f(x) = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} = \frac{p(a)}{q(a)} = f(a)$$

if $q(a) \neq 0$.

Note. If $\lim_{x\to a} f(x)/g(x) = L$ exists and $\lim_{x\to a} g(x) = 0$ then

$$\lim_{x\to a} f(x) = 0$$

For $f(x) = p(x)/q(x)$ if $q(a) = 0$ and $p(a) \neq 0$ then $\lim_{x\to a} f(x)$ does not exist.

If $f(x) = p(x)/q(x)$, $p(x) = q(x) = 0$,

$$p(x) = (x-a)^n p_1(x) \qquad p_1(a) \neq 0$$

$$q(x) = (x-a)^m q_1(x) \qquad q_1(a) \neq 0$$

$$\lim_{x\to a} \frac{p(x)}{q(x)} = \begin{cases} \frac{p_1(a)}{p_2(a)} & \text{if } n = m \\ 0 & \text{if } n > m \\ \text{does not exist} & \text{if } n < m \end{cases}$$

16.4 Example.

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

16.5 Example.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

Let $a \in \mathbb{R}$. What can we say about $\lim_{x\to a} f(x)$? Exists a sequence in $\mathbb{Q}$ that converge to 1, and exists a sequence in $\mathbb{R}\backslash\mathbb{Q}$ that converge to -1. Thus the limit does not exist.

Lecture 17, Oct. 20

Seq Characteriation of Limits

17.1 Theorem. *Let $f(x)$ be defined in an open interval I containing a, except possibly at $x = a$. Then the following are equivalent.*

1. $\lim_{x\to a} f(x) = L$
2. *Whenever $\{x_n\}$ is such that $x_n \to a$ ($x_n \neq a$) we have $f(x) \to L$*

17.2 Example. $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

17.3 Example.

$$g(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$\lim_{x\to 0} g(x) = 0$. In other words, the limit exists (by using squeeze theorem.)

17.4 Example.

$$\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$$

17.5 Example.

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \\ 1 & \text{if } x = 0 \\ \frac{1}{m} & \text{if } x = \frac{k}{m} \in \mathbb{Q} \text{ with } gcd(k,m) = 1 \end{cases}$$

Suppose $\lim_{x\to a} f(x)$ exists. Then the limit is 0 (because for every irrational sequence that approaches a, all element in the irrational sequence is 0.)

17.6 Definition. We say that L is the limit of $f(x)$ from above (from the right) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \epsilon$. We write

$$\lim_{x\to a^+} f(x) = L$$

We say that L is the limit of $f(x)$ from below (from the left) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $-\delta < x - a < 0$, then $|f(x) - L| < \epsilon$. We write

$$\lim_{x\to a^-} f(x) = L$$

Lecture 18, Oct. 24

Midterm: 7:00-8:45

RCH 307 - A-J

RCH 306 - K-O

DWE 3522 - P-W

DWE 3522A - X-Z

Woman in Pure Math/Math Finance Lunch

Tuesday 12:30-1:20 MC5417

18.1 Definition. We say that L is the limit of $f(x)$ from above (from the right) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \epsilon$. We write

$$\lim_{x \to a^+} f(x) = L$$

We say that L is the limit of $f(x)$ from below (from the left) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $-\delta < x - a < 0$, then $|f(x) - L| < \epsilon$. We write

$$\lim_{x \to a^-} f(x) = L$$

Note. Both the Arithmetic Rules and Sequential Characterization hold for one-sided limits. As does the Squeeze Theorem.

$$lim_{x \to a^+} f(x) = L \text{ iff whenever } \{x_n\} \text{ is such that } x_n \to a, a < x_n \text{ we have } \lim_{x \to \infty} f(x_n) = L$$

18.2 Theorem. *The following are equivaent*

1. $\lim_{x \to a} f(x) = L$
2. $\lim_{x \to a^-} f(x) = L$ *and* $\lim_{x \to a^+} f(x) = L$

Proof.

1. Assume that $\lim_{x \to a} f(x) = L$, Let $\epsilon > 0$, then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$. Hence if $0 < x - a < \delta$ then $|f(x) - L| < \epsilon$ and if $0 < a - x < \delta$ then $|f(x) - L| < \epsilon$. Thus $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.
2. Conversely, assume that $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$. Let $\epsilon > 0$. We can find $\delta_1 > 0$ such that if $0 < x - a < \delta_1$ then $|f(x) - L| < \delta$ and $\delta_2 > 0$ such that if $0 < a - x < \delta_1$ then $|f(x) - L| < \delta$. Let $\delta = min\{\delta_1, \delta_2\}$, hence if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

□

18.3 Example.

$$lim_{x \to 0} \frac{|x|}{x}$$

18.4 Example.

$$lim_{x \to 0^+} \sqrt{x} = 0$$

18.5 Definition. A function $f(x)$ is **even** if $f(x) = f(-x)$ for all $x \in \mathbb{R}$ (graph is symmetric about $x = 0$)

Note. If $f(x)$ is even, (assume these limits exist)

$$\lim_{x \to a^+} f(x) = \lim_{x \to -a^-} f(x)$$

$$\lim_{x \to a^-} f(x) = \lim_{x \to -a^+} f(x)$$

In particular, $\lim_{x\to 0} f(x)$ exists iff $\lim_{x\to 0^+} f(x)$ exists iff $\lim_{x\to 0^-} f(x)$ exists.

18.6 Definition. A function $f(x)$ is **odd** if $f(x) = -f(-x)$ for all $x \in \mathbb{R}$ (graph is symmetric about $(0, 0)$)

Note. If $f(x)$ is odd, (assume these limits exist)

$$\lim_{x \to a^+} f(x) = -\lim_{x \to -a^-} f(x)$$

$$\lim_{x \to a^-} f(x) = -\lim_{x \to -a^+} f(x)$$

$\lim_{x\to 0} f(x)$ exists iff $\lim_{x\to 0^+} f(x) = 0$ or $\lim_{x\to 0^-} f(x) = 0$

18.7 Example. $\lim_{x\to 0} \sin x$ and $\lim_{x\to 0} \cos x$

18.8 Example.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

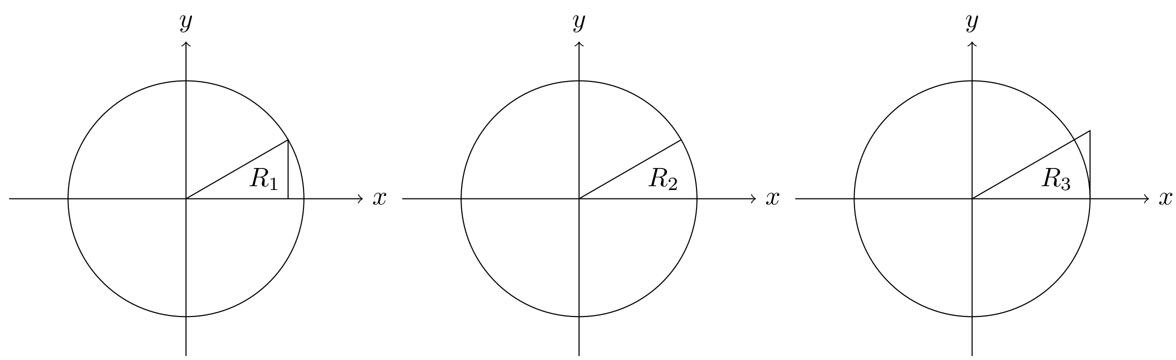
Lecture 19, Oct. 26

Written Assignment 3 Due Wed, Nov. 9

19.1 Theorem (Fundamental Trig Limit)**.**

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Proof. Note that $f(x)$ is even. Hence we need only $\lim_{x\to 0^+}\frac{\sin x}{x}=1$



We have $R_1=\sin x\cos x/2$, $R_2=x/2$ and $R_3=\sin x/(2\cos x)$.

Since $R_1\le R_2\le R_3$, we get

$$\cos x\le\frac{x}{\sin x}\le\frac{1}{\cos x}.$$

Hence

$$\cos x\le\frac{\sin x}{x}\le\frac{1}{\cos x}.$$

By Squeeze Theorem, $\lim_{x\to 0^+}\frac{\sin x}{x}=1$.

□

19.2 Example. Find

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}$$

Solution.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}&=\lim_{x\to 0}\frac{\sin 3x}{3}\cdot\lim_{x\to 0}\frac{4}{\sin 4x}\cdot\frac{3}{4}\\&=1\cdot 1\cdot\frac{3}{4}\\&=\frac{3}{4}\end{aligned}$$

19.3 Example. Find

$$\lim_{x\to 0}\frac{\tan x}{x}.$$

19.4 Example. Find

$$\lim_{x \to 0} \frac{\tan \pi x}{\sin 2x}.$$

Asymptotes and Limits at ∞

19.5 Definition. We say that L is the limit as x approaches infinity of $f(x)$ if for every $\epsilon > 0$, there exists $M > 0$ such that if $x \geq M$, then $|f(x) - L| < \epsilon$. We write

$$\lim_{x \to \infty} f(x) = L.$$

19.6 Example. If $f(x) = 1/x$, then $\lim_{x \to \infty} f(x) = 0$.

Note. Arithmetic Rules, Sequential Characterization and Squeeze Theorem carry through.

19.7 Theorem (Fundamental Log Limit)**.**

$$\lim_{x \to \infty} \frac{ln(x)}{x} = 0$$

Proof.

$$\frac{ln(x)}{x} = \frac{2ln(x^{1/2})}{x^{1/2} \cdot x^{1/2}} = \frac{2ln(x^{1/2})}{x^{1/2}} \cdot \frac{1}{x^{1/2}} < \frac{2}{x^{1/2}}$$

By squeeze theorem, $\lim_{x \to \infty} \frac{ln(x)}{x} = 0$ □

19.8 Example. Find

$$\lim_{x \to \infty} \frac{ln(x)}{x^{1/100}}$$

Note.

$$\lim_{x \to \infty} \frac{ln(x)}{x^p} = 0 \text{ if } p > 0$$

19.9 Example.

$$\lim_{x \to \infty} \frac{x^p}{e^x} = 0$$

Lecture 20, Oct. 27

20.1 Definition. $L = \lim_{x\to\infty} f(x)$ if for every $\epsilon > 0$, there exists $M > 0$ such that $x \geq M$, then

$$|f(x) - L| < \epsilon.$$

20.2 Example.
1. If $p > 0$, we have $\lim_{x\to\infty} \frac{1}{x^p} = 0$
2. $\lim_{x\to\infty} \frac{lnx}{x} = 0$

Variants

1. If $p > 0$, we have $\lim_{x\to 0} \frac{lnx}{x^p} = 0$
2. For all p, $\lim_{x\to 0} \frac{(lnx)^p}{x} = 0$
3. $\lim_{x\to\infty} \frac{x}{e^x} = 0$

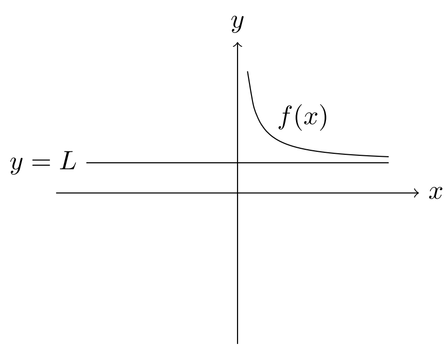
20.3 Definition. We say that L is the limit of $f(x)$ as x approaches $-\infty$ if for every $\epsilon > 0$ there exists $M > 0$ such that if $x < -M$, then $|f(x) - L| < \epsilon$. We write

$$\lim_{x\to-\infty} f(x) = L.$$

20.4 Example. By Squeeze Theorem, we have

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

20.5 Definition (Asymptote)**.** Assume $\lim_{x\to\pm\infty} f(x) = L$, then the line $y = L$ is called a horizontal asymptote of f(x).



Infinite Limits

20.6 Definition. We say that $f(x)$ approaches ∞ at $x = a$ if for every $M > 0$ there exists $\delta > 0$ such that if $|x - a| < \delta$, then $f(x) > M$. We write

$$\lim_{x\to a} f(x) = \infty$$

20.7 Definition (Vertical Asymptote)**.** If $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$, then $x = a$ is called a vertical asymptote for $f(x)$

Lecture 21, Oct. 28

EA 3 due Fri Nov. 4

WA 3 due Wed Nov. 9

21.1 Definition (Continuity)**.** We say that $f(x)$ is **continuous** at $x = a$ if

1. $\lim_{x \to a} f(x)$ exists
2. $\lim_{x \to a} f(x) = f(a)$

Equivalently, we say that $f(x)$ is continuous at $x = a$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$.

If $f(x)$ is not continuous at $x = a$ we say that f is **discontinuous** at $x = a$. We write

$$D(f) = \{a \in \mathbb{R} \mid f \text{ is discontinuous at } x = a\}$$

21.2 Theorem (Sequential Characterization of Limit)**.** *Assume that $f(x)$ is defined on an open interval I containing $x = a$, Then the following are equivalent:*

1. *$f(x)$ is continuous at $x = a$*
2. *If $\{x_n\}$ with $x_n \to a$, we have $f(x_n) \to f(a)$*

Proof. Assume that $f(x)$ is continuous at $x = a$. Let $\{x_n\}$ be such that $x_n \to a$. Let $\epsilon > 0$. Since $f(x)$ is continuous at $x = a$, there exists a $\delta > 0$ such that for all $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. Since $\{x_n\}$ converges to a, there exists a $N_0 > 0$ such that for all $n > N_0$ we have $|x_n - a| < \delta$. Then if $n \geq N_0$, we have $|f(x_n) - f(a)| < \epsilon$.

Conversely, for a contraposition, that $f(x)$ is not continuous at $x = a$. Then there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ there exists $x_\delta \in (a - \delta, a + \delta)$ with $|f(x_\delta) - f(a)| \geq \epsilon_0$. In particular, there exists a $x_n \in (a - \frac{1}{n}, a + \frac{1}{n})$ with $|f(x_n) - f(a)| > \epsilon_0$. Hence $f(x_n)$ does not converge to $f(a)$. $\square$

21.3 Theorem (Arithmetic Rules)**.** *Assume $f(x)$ and $g(x)$ are continuous at $x = a$, then*

1. *$(cf)(x)$ is continuous at $x = a$ for $c \in \mathbb{R}$*
2. *$(f + g)(x)$ is continuous at $x = a$*
3. *$(fg)(x)$ is continuous at $x = a$*
4. *$(f/g)(x)$ is continuous at $x = a$ provided that $g(a) \neq 0$.*

21.4 Question. Let $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$. Let $h(x) = g \circ f(x) = g(f(x))$. Assume that $\lim_{x \to a} f(x) = L$ and $\lim_{y \to L} g(y) = M$.

Is $\lim_{x \to a} g \circ f(x) = \lim_{x \to a} h(x) = M$?

21.5 Theorem. *If $f(x)$ is continuous at $x = a$, and $g(y)$ is continuous at $y = f(a)$, then $h(x) = g \circ f(x)$ is continuous at $x = a$.*

Proof. Let $x_n \to a$, then $f(x_n) \to f(a)$, hence $g(f(x_n)) \to g(f(a))$ ☐

21.6 Example. Show that $\sin x$ is continuous.

Observation:

1. $\sin x$ is continuous at $x = 0$ since $\lim_{x \to 0} \sin x = 0$.
2. If we can show that $\lim_{h \to 0} \sin(x_0 + h) = \sin x_0$ then $\sin x$ is continuous at x_0.

$$\begin{aligned}\lim_{h \to 0} \sin(x_0 + h) &= \lim_{h \to 0} [\sin x_0 \cos h + \sin h \cos x_0] \\ &= \sin x_0\end{aligned}$$

Nature of Discontinuity

21.7 Example.

$$f(x) = \frac{x^2 - 1}{x - 1}$$

$f(x)$ is not continuous at $x = 1$.

Let

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

21.8 Definition. If $\lim_{x \to a} f(x) = L$ exists but $L \neq f(a)$, then we say that $f(x)$ has a **removable discontinuity** at $x = a$. Let

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ L & \text{if } x = a \end{cases}$$

21.9 Definition. If $\lim_{x \to a} f(x)$ does not exists, then $x = a$ is called an **essential discontinuity** for $f(x)$.

3 Types of Essential Discontinuities

1. Finite jump discontinuity: $\lim_{x \to a^+} f(x) = L$, $\lim_{x \to a^-} f(x) = M$ and $L \neq M$
2. Vertical Asymptote: $\lim_{x \to a^\pm} f(x) = \pm\infty$
3. Oscillatory Discontinuity: $\lim_{x \to 0} \sin(1/x)$

Lecture 22, Oct. 31

Anton's tutorial on Tuesady is cancelled.

Aside

Suppose $f\colon \mathbb{R} \to \mathbb{R}$.

$$D(f) = \{x_0 \in \mathbb{R} \mid f \text{ is discontinuous at } x_0\}$$

$$D_n(f) = \{x_0 \in \mathbb{R} \mid \forall \delta > 0\, \exists x, y \in (x_0 - \delta, x_0 + \delta)\ \ |f(x) - f(y)| \geq \frac{1}{n}\}$$

Then if $x_0 \in D_n(f)$ for some n, then $x_0 \in D(f)$.

Note.

$$D(f) = \bigcup_{n=1}^{\infty} D_n(f)$$

22.1 Definition. A set $A \subset \mathbb{R}$ is called F_σ if

$$A = \bigcup_{n=1}^{\infty} F_n$$

where each F_n is closed.

22.2 Example. Let $\mathbb{Q} = \{r_1, r_2, r_3, \cdots\}$

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\} \to F_\sigma$$

22.3 Definition. A set $A \subset \mathbb{R}$ is called G_δ if

$$A = \bigcap_{n=1}^{\infty} U_n$$

where each U_n is open.

Note.

1. $\{0\} = \bigcup_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})$
2. A is G_δ iff A^c is F_σ
3. $D(f)$ is $F_\sigma \to D(f)^c$ is G_δ

Note. $\mathbb{Q} = \{r_1, r_2, r_3, \cdots\}$

$$U_k = \bigcup_{n=1}^{\infty} (r_n - \frac{1}{2^{n+k+1}}, r_n + \frac{1}{2^{n+k+1}}) \supset \mathbb{Q}$$

So is it true that

$$\bigcap_{k=1}^{\infty} U_k = \mathbb{Q}$$

Continuity on an Interval

22.4 Question. Is $f(x) = \sqrt{x}$ continuous at $x = 0$?

22.5 Definition (Continuity on an Interval)**.** We say that $f(x)$ is continuous on the open interval (a, b) if $f(x)$ is continuous at each $x_0 \in (a, b)$

We say that $f(x)$ is continuous on the closed interval $[a, b]$ if $f(x)$ is continuous at (a, b) and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

Similarly for $(a, b], (a, \infty), \cdots$

22.6 Theorem (Sequential Characterization for Continuity on [a,b])**.** *Let $f\colon [a, b] \to \mathbb{R}$, then the followings are equivalent:*

1. *f is continuous at $[a, b]$*
2. *if $\{x_n\} \subset [a, b]$ with $x_n \to x_0 \in [a, b]$ then $f(x_n) \to f(x_0)$*

Remark. Given $S \subset \mathbb{R}, S \neq \emptyset$, we say that $f\colon S \to \mathbb{R}$ is continuous on S is whenever $\{x_n\}$ is a sequence in S with $x_n \to x_0 \in S$ we have $f(x_n) \to f(x_0)$

mathmode inline: $x = 0$ $x = 0$ display:

$$x = 0$$

$$x = 0$$

$$x = 0$$

$$x = 0 \tag{1}$$

Lecture 23, Nov. 2

23.1 Theorem (Intermediate Value Thm (IVT))**.** *If $f(x)$ is continuous on $[a,b], f(a) < 0$ and $f(b) > 0$, then there exists $c \in (a,b)$ with $f(c) = 0$.*

Proof. Let $E = \{x \in [a,b] \mid f(x) \leq 0\}$. Then $E \neq \emptyset$ since $a \in E$. Since E is bounded, it has a lub which we call c (Note: $c \in [a,b]$). We claim $f(c) = 0$ We can find $x_n \in E$ with $x_n \to c$. By the Sequential Characterization of Continuity $f(x_n) \to f(c)$. Since $f(x_n) \leq 0$ for all n, $f(c) \leq 0$. Observe that $c < B$ for each $n \in \mathbb{N}$. We choose $y_n \in [a,b]$ so that $c < y_n \leq b$ and $|c - y_n| < \frac{1}{n}$. Since $y_n \to c$, we have $f(y_n) \to f(c)$. But $f(y_n) > 0$ for all n, so $f(c) \geq 0$. Thus $f(c) = 0$. □

Note. A similar statement holds if $f(a) > 0$ and $f(b) < 0$.

23.2 Corollary (Intermediate Value Theorem II)**.** *If $f(x)$ is continuous on $[a,b]$, and if $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$, then there exists $c \in (a,b)$ with $f(c) = \alpha$.*

Proof. Let $g(x) = f(x) - \alpha$ and apply the theorem 23.1. □

23.3 Question. Assume $f\colon [a,b] \to \mathbb{R}$ is 1-1. What can we say about f if $f(x)$ is also continuous? Is f strictly monotonic?

23.4 Definition. We say that $f(x)\colon [a,b]$ is non-decreasing on $[a,b]$ if whenever $x, y \in [a,b]$ with $x < y$, we have $f(x) \leq f(y)$. We say that f is strictly increasing if whenever $x, y \in [a,b]$ with $x < y$ we have $f(x) < f(y)$. Similarly we could define non-increasing and strictly decreasing.

f is monotonic on $[a,b]$ if it is either non-decreasing or non-increasing. f is strictly monotonic if it is strictly increasing or strictly decreasing.

23.5 Corollary.

TO BE FINISHED

Lecture 24, Nov. 4

24.1 Theorem (Intermediate Value Thm (IVT))**.** *If $f(x)$ is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists $c \in (a, b)$ with $f(c) = 0$.*

24.2 Example. Show that

$$f(x) = x^2 + x - 3$$

has a root on $[0, 4]$.

Solution. Observation:

1. $f(x)$ is continuous.
2. $f(0) = -3 < 0$
3. $f(4) > 0$

By the theorem 24.1 there exists $c \in [0, 4]$ with $f(c) = 0$.

24.3 Question. How do we find c?

Solution. Binary Search Algorithm.

Set Up $f(x)$ is continuous. We want to solve $f(x) = 0$.

Step 1 Find $a < b$ with $f(a)f(b) < 0$.

Algorithm

1. $a_1 = a, b_1 = b$
2. If $|b - a| < 2\epsilon$, let $d = \frac{a+b}{2}$, stop
3. If $f(d) = 0$, stop
4. If $f(a)f(d) < 0$, let $a_1 = a_1$, $b_1 = d$, goto 1.
5. If $f(d)f(b) < 0$, let $a_1 = d$, $b_1 = b_1$, goto 2

24.4 Example. Show that there exists $c \in [0, \frac{\pi}{2}]$ with $\cos c = c$

Solution. Let $f(x) = \cos x - x$, which is continuous on $[0, \frac{\pi}{2}]$. Observe that $f(0) > 0$ and $f(\frac{\pi}{2}) < 0$. By the theorem 24.1 there exists c with $f(c) = 0 = \cos c - c$.

24.5 Theorem (Extreme Value Theorem)**.** *If $f(x)$ is continuous on $[a, b]$, then there exists $c, d \in [a, b]$ such that*

$$f(c) \leq f(x) \leq f(d)$$

for all $x \in [a, b]$.

Proof. First we show that $f(x)$ is bounded. Suppose it is not bounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ with $f(x_n) \geq n$. By the BWT, $\{x_n\}$ has a convergent sub-sequence $\{x_{n_k}\}$ with $x_{n_k} \to x_0 \in [a, b]$. Since f is continuous, $f(x_{n_k}) \to f(x_0)$. But $f(x_{n_k}) \geq n_k \to \infty$, which is impossible. Thus $f([a, b])$ is bounded.

Let $M = lub(f([a, b]))$. For each $n \in \mathbb{N}$ choose $y_m \in [a, b]$ with $M - \frac{1}{n} < f(y_n) \leq M$. By the BWT, $\{y_n\}$ has a convergent sub-sequence $\{y_{n_k}\}$ with $y_{n_k} \to d \in [a, b]$. Hence $f(d) = \lim_{k \to \infty} f(x_{n_k}) = M$ $\square$

Lecture 25, Nov. 7

25.1 Theorem (Extreme Value Theorem)**.** *If $f(x)$ is continuous on $[a, b]$, then there exists $c, d \in [a, b]$ such that*

$$f(c) \le f(x) \le f(d)$$

for all $x \in [a, b]$.

Uniform Continuity

25.2 Question. Assume that $f(x)$ is continuous on some interval I. Let $\epsilon > 0$. Does there exists a single $\delta > 0$ such that for every $a \in I$, we have if $|x - a| < \delta$, $x \in I$, then $|f(x) - f(a)| < \delta$?

25.3 Definition (Uniform Conitnuity)**.** We say that $f(x)$ is uniformly continuous on $S \subset \mathbb{R}$ if for every ϵ, there exists a $\delta > 0$ such that if $|x - y| < \delta$, $x, y \in S$, then $|f(x) - f(y)| < \delta$.

25.4 Theorem (Sequential Characterization for Uniform Continuity)**.** *Let $f \colon S \to \mathbb{R}$. Then the followings are equivalent*

1. *$f(x)$ is continuous on S*
2. *If $\{x_n\}, \{y_n\} \subset S$ with $\lim_{x\to\infty} |x_n - y_n| = 0$, then $\lim_{n\to\infty} |f(x_n) - f(y_n)| = 0$.*

Proof. Assume that $f(x)$ is uniformly continuous on S. Let $\epsilon > 0$ and let $\{x_n\}, \{y_n\} \subset S$ with $|x_n - y_n| \to 0$. Choose $\delta > 0$ so that if $x, y \in S$, $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. We can pick $N_0 \in \mathbb{N}$ so that if $n \ge N_0$, then $|x_n - y_n| < \delta$. It follows that if $n \ge N_0$, then $|f(x_n) - f(y_N)| < \epsilon$. Hence $\lim_{x\to\infty} |f(x_n) - f(y_n)| = 0$.

Conversely, assume that 1 fails ($f(x)$ is not uniformly continuous on S). Then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ we can find $x_\delta, y_\delta \in S$ with $|x_\delta - y_\delta| < \delta$, but $|f(x_\delta) - f(y_\delta)| \ge \epsilon_0$. Let $\delta = 1/n$, and $x_\delta = x_n$, $y_\delta = y_n$. This gives us $\{x_n\}, \{y_n\} \subset S$, with $|x_n - y_n| < 1/n \to 0$, but $\lim_{x\to\infty} |f(x_n) - f(y_n)| \ne 0$ ☐

25.5 Theorem. *If $f(x)$ is continuous on $[a, b]$, then $f(x)$ is uniformly continuous on $[a, b]$.*

Proof. Assume that $f(x)$ is not uniformly continuous on $[a, b]$, then there exists ϵ_0 and $\{x_n\}, \{y_n\} \subset S$ with $|x_n - y_n| \to 0$, but $|f(x_n) - f(y_n)| \ge \epsilon_0$ for all n.

By the BWT $\{x_n\}$ has a sub-sequence $\{x_{n_k}\}$ with $x_{n_k} \to a \in S$. Since $|x_{n_k} - y_{n_k}| \to 0$, then $y_{n_k} \to a$. But then $\lim_{x\to\infty} |f(x_{n_k}) - f(y_{n_k})| = 0$, which is impossible. ☐

Lecture 26, Nov. 9

26.1 Theorem. *If $f(x)$ is continuous on $[a, b]$, then f is uniformly continuous.*

Basic Facts about Uniform Continuity

1. If f is uniformly continuous on $S \subset \mathbb{R}$ and if $T \subseteq S$ then f is uniformly continuous on T.
2. If f is uniformly continuous on S and if $\{x_n\} \subset S$ is Cauchy then $\{f(x_n)\}$ is Cauchy.
3. If f is uniformly continuous on (a, b), then $\lim_{x\to a^+} f(x)$ exists and $\lim_{x\to b^-} f(x)$ exists.
4. f is uniformly continuous on (a, b) iff there exists $F\colon [a, b] \to R$ such that F is continuous on $[a, b]$ and $F(x) = f(x)$ for all $x \in (a, b)$.
5. if $f(x)$ is uniformly continuous on (a, b), then $f((a, b))$ is bounded.

Derivatives

26.2 Definition (Differentiable)**.** We say that f is differentiable at $x = a$ if

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists. In this case, we write

$$f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

and we call $f'(x)$ the derivative of f at $x = a$.

26.3 Definition (Tangent Line)**.** Assume that $f'(x)$ exists. Then the line with slope $f'(x)$ passing through $(a, f(a))$ is called the tangent line to $f(x)$ at $x = a$.

$$y = f(a) + f'(a(x - a))$$

26.4 Definition (Alternative Definition)**.**

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$

26.5 Example. $f(x) = \cos x$, find $f'(0)$

Solution.

$$\begin{aligned}
f'(0) &= \lim_{h\to 0} \frac{\cos h - \cos 0}{h} \\
&= \lim_{h\to 0} \frac{\cos h - 1}{h} \\
&= \lim_{h\to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos x + 1)} \\
&= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{-\sin h}{h} \frac{\sin h}{(\cos h + 1)} \\
&=0
\end{aligned}$$

26.6 Example. $f(x) = \sin x$, find $f'(a)$

Solution.

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{\sin(a+h) - \sin(a)}{h} \\ &= \lim_{h\to 0} \frac{\sin a \cos h + \cos a \sin h - \sin a}{h} \\ &= \lim_{h\to 0} \sin a \frac{\cos h - 1}{h} + \cos a \frac{\sin h}{h} \\ &= \cos a \end{aligned}$$

26.7 Theorem. *If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.*

Proof. Since $\lim_{x\to a}(f(x) - f(a))/(x-a)$ exists and $\lim_{x\to a} x - a = 0$, we have $\lim_{x\to a} f(x) - f(a) = 0 \iff \lim_{x\to a} f(x) = f(a)$. □

26.8 Example.

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is $g(x)$ differentiable at $x = 0$?

26.9 Example.

$$h(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is $h(x)$ differentiable at $x = 0$?

Lecture 27, Nov. 10

27.1 Theorem (Arithmetic Rules for Differentiation)**.** *Assume that $f(x)$, $g(x)$ are differentiable at $x = a$.*

1. *If $f(x) = c$ for all x, then $f'(a) = 0$*
2. *$(f + g)(x)$ is differentiable at $x = a$ with $(f + g)'(a) = f'(a) + g'(a)$*
3. *$(fg)(x)$ is differentiable at $x = a$ with $(fg)'(a) = f'(a)g(a) + g'(a)f(a)$*
4. *Let $h(x) = \frac{1}{f(x)}$. Then $h(x)$ is differentiable at $x = a$ if $f(a) \neq 0$ and*
$$h'(a) = \frac{-f'(a)}{f(a)}$$
5. *If $h(x) = \frac{f(x)}{g(x)}$ then $h(x)$ is differentiable at $x = a$, if $g(a) \neq 0$ and*
$$h'(a) = \frac{f'(a)g(a) - g'(a)f(a)}{g^2(a)}$$

Proof.

3 Consider
$$\lim_{x\to a} \frac{(fg)(x) - (fg)(a)}{x - a}$$

$$\begin{aligned}
&\lim_{x\to a} \frac{(fg)(x) - (fg)(a)}{x - a} \\
=&\lim_{x\to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a} \\
=&\lim_{x\to a} g(x)\frac{f(x) - f(a)}{x - a} + \lim_{x\to a} f(a)\frac{g(x) - g(a)}{x - a} \\
=&\lim_{x\to a} g(x) \lim_{x\to a} \frac{f(x) - f(a)}{x - a} + f(a)g'(a) \\
=&g(a)f'(a) + f(a)g'(a)
\end{aligned}$$

4 Consider
$$\lim_{x\to a} \frac{1/f(x) - 1/f(a)}{x - a}.$$

$$\begin{aligned}
&\lim_{x\to a} \frac{1/f(x) - 1/f(a)}{x - a} \\
=&\lim_{x\to a} \frac{f(a) - f(x)}{x - a} \cdot \frac{1}{f(a)f(x)} \\
=&\frac{-f'(a)}{f^2(a)}
\end{aligned}$$

5 Combine 3 and 4.

$\square$

Linear Approximation

Note. Assume that $f(x)$ is differentiable at $x = a$. Then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

If $x \approxeq a$, then

$$\begin{aligned} f'(a) &\approxeq \frac{f(x) - f(a)}{x - a} \\ \Rightarrow f'(a)(x - a) &\approxeq f(x) - f(a) \\ \Rightarrow f(x) &\approxeq f'(a)(x - a) + f(a) \end{aligned}$$

27.2 Definition. Let $f(x)$ be differentiable at $x = a$. We define the linear approximation to $f(x)$ at $x = a$ to be the function

$$L_a^f(x) = f(a) + f'(a)(x - a)$$

27.3 Theorem (Properties of Linear Approximation)**.** $L_a^f(x)$ *has the following properties*

1. $L_a^f(a) = f(a)$
2. $(L_a^f)'(x) = f'(a)$
3. *If* $h(x) = mx + b$ *and* $h(x)$ *satisfies 1) and 2) then* $h(x) = L_a^f(a)$
4. $L_a^f(a) \approxeq f(x)$ *if* $x \approxeq a$
5. *The graph of* $L_a^f(a)$ *is the tangent line to graph of* $f(x)$ *at* $x = a$

27.4 Example. Consider $f(x) = \sin x$.

$$L_0^{\sin x} = \sin 0 + \cos 0(x - 0) = x$$

27.5 Example. Consider $f(x) = e^x$, we have $f(0) = 1$ and $f'(0) = 1$. Then

$$L_0^{e^x} = f(0) + f'(0)(x - 0) = 1 + x$$

27.6 Example. If $f(x) = e^{-u^2}$,

$$e^{-u^2} \approxeq 1 - u^2$$

if u is small.

Lecture 28, Nov. 11

Chain Rule Assume $f\colon I \to \mathbb{R}$, with I open and containing $x = a$, $g\colon J \to \mathbb{R}$, with J open and containing $f(a)$ with $f(I) \subset J$. Assume that f is differentiable at $x = a$ and g is differentiable at $y = f(a)$. Let $h(x)\colon I \to \mathbb{R}$ be $h(x) = g \circ g(x) = g(f(x))$.

28.1 Question. Is $h(x)$ differentiable at $x = a$ and if so what is $h'(a)$?

We know that if $x \approxeq a$, then

$$f(x) \approxeq L_a^f(x)$$

and if $y \approxeq f(a)$ then

$$g(y) = L_{f(a)}^g(y).$$

If $x \approxeq a$, then $f(x) \approxeq f(a)$, hence $h(x) = g(f(x)) \approxeq g(L_a^f(x)) \approxeq L_{f(a)}^g(L_a^f(x))$.

The equation

$$\begin{aligned} & L_{f(a)}^g \circ L_a^f(x) \\ =& L_{f(a)}^g(f(a) + f'(a)(x-a)) \\ =& g(f(a)) + g'(f(a))(f(a) + f'(a)(x-a) - f(a)) \\ =& g(f(a)) + g'(f(a))f'(a)(x-a) \\ =& h(a) + h'(a)(x-a) \\ =& L_a^h(x) \end{aligned}$$

holds if an only if $h'(x) = g'(f(a))f'(a)$.

28.2 Theorem (Chain Rule). *If $f\colon I \to \mathbb{R}$ is an open interval containing $x = a$, $g\colon J \to \mathbb{R}$ is an open interval containing $y = f(a)$, $f(I) \subset J$, J is differentiable at $x = a$, g is differentiable at $y = f(a)$, then if $h\colon I \to \mathbb{R}$ is given by $h(x) = g \circ f(x)$, then h is differentiable at $x = a$ with $h'(a) = g'(f(a))f'(a)$.*

False proof.

$$\begin{aligned} h'(a) &= \lim_{x\to a} \frac{h(x) - g(a)}{x-a} \\ &= \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{x-a} \\ &= \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x-a} \\ &= \lim_{y\to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} \cdot \lim_{x\to a} \frac{f(x) - f(a)}{x-a} \\ &= g'(f(a)f'(a) \end{aligned}$$

This is false because $f(x)$ might equal $f(a)$, and thus you multiplied $\frac{0}{0}$. ☐

Real Proof. Let

$$\varphi(y) = \begin{cases} \frac{g(y)-g(f(a))}{y-f(a)} & \text{if } y \neq f(a) \\ g'(f(a)) & \text{if } y = f(a) \end{cases}$$

Note that $\varphi(y)$ is continuous.

Observe that $g(y) - g(f(a)) = \varphi(y)[y - f(a)]$ for all $y \in J$, even $y = f(a)$. Then now,

$$
\begin{aligned}
h'(a) &= \lim_{x \to a} \frac{h(x) - g(a)}{x - a} \\
&= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} \\
&= \lim_{x \to a} \frac{\varphi(f(x))[f(x) - f(a)]}{x - a} \\
&= \lim_{x \to a} \varphi(f(x)) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \varphi(f(a)) \cdot f'(a) \\
&= g'(f(a)) f'(a)
\end{aligned}
$$

□

28.3 Example. Consider $h(x) = \cos x = \sin(x + \pi/2)$

Lecture 29, Nov. 14

29.1 Definition. Assume that $f(x)$ is differentiable at each x_0 in an interval I. We define $f' : I \to R$ by

$$f'(x) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

. f' is called the derivative (function) of f on I.

29.2 Example. $f(x) = sinx$, $f'(x) = cosx$ on $\mathbb{R}$

Notation.

1. $y = f(x) \to y'$ will denote $f'(x)$
2. $\frac{dy}{dx} = f'(x)$
3. $\frac{d}{dy} f(x) = f'(x)$

If $f'(x)$ is differentiable at $x_0 \in \mathbb{R}$, then we call

$$(f')'(x_0) = \lim_{h \to 0} \frac{f'(x_0 + h) - f'(h)}{h}$$

the second derivative of f at $x = x_0$. We denote this by $f''(x_0)$.

In general if f is twice differentiable on I, we write $f''(x)$ to represent the second derivative.

$f'''(x) \to$ third derivative.

$f^{(n)}(x)$ denotes the n-th derivative.

29.3 Theorem (More on Linear Approximation)**.** *If $f(x)$ is differentiable at $x = a$, and if*

$$L_a(x) = f(a) + f'(a)(x - a)$$

then $L_a(x) \approxeq f(x)$ if $x \approxeq a$

29.4 Theorem (Error in Linear Approximation)**.**

$$Error = |f(x) - L_a(x)|$$

The error is effected by

1. *Distance of x to a.*
2. *The larger $|f''(x)|$ the larger the error may be.*

Both 1 and 2 hold in general most of the time but not always.

29.5 Theorem (Newton's Method)**.** *Pick a_1.*

Let

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$$

Remark.

1. If $f'(c) \neq 0$, then there exists $\delta > 0$ such that if $a_1 \in (c - \delta, c + \delta)$ then $a_n \to c$
2. When the method work, the convergence is generally very fast. In general, the convergence is "quadratic" in nature. Roughly speaking this means the number of decimal places of accuracy will at least double with each iteration.
3. It can fail.

29.6 Example (Heron's Method)**.** Solve $x^2 - a = 0$.

Solution. Pick a_1.

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} = a_n - \frac{a_n^2 - a}{2a_n} = \frac{1}{2}(a_n + \frac{a}{a_n})$$

Lecture 30, Nov. 16

Maxima, Minima and Critical Points

30.1 Definition (Global Maximum and Minimum)**.** Let f be defined on an interval I. We say that, $d \in I$ is a global maximum for f on I if

$$f(x) \leq f(d) \text{ for all } x \in I$$

and $f(d)$ is the global maximum value.

Similarly we define the global minimum and global minimum value.

30.2 Example. $f(x) = x$ on $(0, 1)$ has no global maximum or minimum on $(0, 1)$.

30.3 Definition (Local Maximum and Minimum)**.** We say that c ks a local maximum for $f(x)$ if there exists an open interval (a, b) containing c with

$$f(x) \leq f(c) \text{ for all } x \in (a, b)$$

Similarly we define the local minimum.

30.4 Theorem (The Might-be-on-the-exam Theorem)**.**

1. *Assume that $f(x)$ has a local maximum at $x = c$. If $f(x)$ is differentiable at $x = c$ then $f'(c) = 0$*
2. *Assume that $f(x)$ has a local minimum at $x = c$. If $f(x)$ is differentiable at $x = c$ then $f'(c) = 0$. (Might be on the exam)*

Proof.

1. Since $x = c$ is a local maximum for $f(x)$ there exists $\delta > 0$ such that if $c - \delta < x < c + \delta$, then $f(x) \leq f(c)$. Then if $c - \delta < x < c$,
$$\frac{f(x) - f(c)}{x - c} \geq 0$$
and if $c < x < c + \delta$,
$$\frac{f(x) - f(c)}{x - c} \leq 0$$
Thus
$$\frac{f(x) - f(c)}{x - c} = 0$$
Hence $f'(c) = 0$. □

30.5 Definition (Critical Point)**.** Assume that f is defined on an open interval I. We call $c \in I$ a critical point for f if either

1. $f'(c) = 0$
2. f is not differentiable at $x = c$.

Note. Given f continuous on $[a.b]$, then the global max (min) will be at

1. either $x = a$ or $x = b$ or
2. a critical point in (a, b).

Lecture 31, Nov. 17

Inverse Function Theorem

Note. If f is $1-1$, we get $f\colon X \to range(f) \subset Y = \{y \in Y \mid y = f(x) \text{ for some } x\}$. If f is $1-1$ and onto its range, we can define $g\colon range(f) \to x$ by $g(y) = x$ if and only if $f(x) = y$.

31.1 Definition. We say that f is invertible on $A \subset R$ if f is $1-1$ on A. In this case, we define the inverse of f on A by

$$g(y) = x \iff y = f(x) \text{ for } x \in A$$

Note. Geometrically the inverse function is the reflection of the original function through $y = x$.

31.2 Example. $f(x) = mx + b$ is always invertible if $m \neq 0$. The inverse function is

$$g(y) = \frac{1}{m}x - \frac{b}{m}$$

Observation. We have

$$L^g_{f(a)}(x) = \frac{1}{f'(a)}(x - f(a))$$

$$g'(f(a)) = \frac{1}{f'(a)}$$

31.3 Definition. We say that $f(x)$ is increasing (strictly increasing) on an interval I if whenever $x_1, x_2 \in I$ with $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$ $(f(x_1) < f(x_2))$.

Similarly we define "decreasing (strictly decreasing)".

We say that f is monotonic on I if one of these holds.

Basic Facts.

1. If $f(x)$ is strictly increasing or decreasing on I, then f is $1-1$ on I, and hence invertible on I.
2. If f is continuous on I and $1-1$ then f is either strictly increasing or strictly decreasing.
3. Assume that $f(x)$ is increasing on $[a, b]$. Let $c \in (a, b)$. Claim that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists with $\lim_{x\to a^-} f(x) \leq \lim_{x\to a^+} f(x)$

31.4 Theorem. *Assume that $f(x)$ is increasing on $[a, b]$, then the following are equivalent*

1. *$f(x)$ is continuous on $[a, b]$*
2. *$f([a, b]) = [f(a), f(b)]$*

Lecture 32, Nov. 18

32.1 Theorem. *If $f\colon [a,b] \to \mathbb{R}$ is increasing, then TFAE*

1. *$f(x)$ is continuous on $[a,b]$*
2. *$f([a,b]) = [f(a), f(b)]$*

32.2 Corollary. *If $f\colon [a,b]$ is strictly monotonic with inverse $g\colon f([a,b]) \to [a,b]$ then f is continuous on $[a,b]$ if and only if g is continuous on $f([a,b])$.*

32.3 Theorem (Inverse Function Theorem)**.** *Assume that if $f\colon [a,b] \to \mathbb{R}$ is strictly monotonic with inverse $g\colon f([a,b]) \to \mathbb{R}$. If f is continuous on $[a,b]$, differentiable on $[a,b]$, and if $x_0 \in (a,b)$ with $f'(x_0) \neq 0$ with $y_0 = f(x_0)$, then g is differentiable at y_0 with*

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}.$$

Proof. Let $\{y_n\} \subset f([a,b])$ with $y_n \to y_0$, $y_n \neq y_0$. Let $x_n = g(y_n) \in [a,b]$. Since f and g are continuous, $g(y_n) \to g(y_0) \Rightarrow x_n \to x_0$. Then

$$\lim_{x\to\infty} \frac{g(y_n) - g(y_0)}{y_n - y_0} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n - f(x_0))} = \lim_{n\to\infty} \frac{1}{f'(x_0)}.$$

By the Sequenctial Characterization of limits $g'(x) = \lim_{n\to\infty} \frac{1}{f'(x_0)}$ ☐

32.4 Example. $f(x) = x^3$ and $g(x) = x^{1/3}$.

$$f'(0) = 0$$

$$g'(x) = \begin{cases} \frac{1}{3x^{2/3}} & \text{if } x \neq 0 \\ \text{does not exist} & \text{if } x = 0 \end{cases}$$

32.5 Example (Inverse Trig Functions)**.**

1. $\arcsin x$

   $f(x) = \sin x$ on $[-\pi/2, \pi/2]$, $f(x)$ is strictly increasing $\Rightarrow$ invertible on $[-\pi/2, \pi/2]$.

   $$\sin([-\pi/2, \pi/2]) = [-1, 1].$$

   Define $g(x) = \arcsin(y)$ on $[-1,1]$ by $g(y) = x$ iff $\sin x = y$ for $x \in [-\pi/2, \pi/2]$
   If $g(y) = \arcsin y$. if $y_0 \in (-1,1)$,

   $$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{\cos x}.$$

   where $f(x) = \sin x$ and $x_0 = \arcsin y_0$ and $y_0 = \sin x_0$, $x_0 \in (-\pi/2, \pi/2)$. Since $\cos x_0 = \sqrt{1 - \sin^2 x_0} = \sqrt{1 - y_0^2}$,

   $$g'(y_0) = \frac{1}{\sqrt{1 - y_0^2}}.$$

   Note. $\sin(\arcsin x) = x$ holds for $x \in [-1,1]$ while $\arcsin(\sin x) = x$ Holds iff $x \in [-\pi/2, \pi/2]$

2. $\arctan x$

   For each $y \in \mathbb{R}$ define $g(y) = \arctan y$ by $g(y) = x$ iff $\tan x = y$ for $x \in (-\pi/2, \pi/2)$. That is,

$$\arctan y : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$$

   with $\tan(\arctan y) = y$ for $y \in \mathbb{R}$

   Note that

$$\frac{d}{dx} \tan x = \sec^2 x = \frac{1}{\cos^2 x}$$

   By the Inverse Function Theorem,

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{\sec^2(\arctan y)} = \frac{1}{1 + \tan^2(\arctan y)} = \frac{1}{1 + y^1}$$

3. $\arccos y$ $\cos(x)$ is 1-1 on $[0, \pi]$

   Note. $\cos([0, \pi]) = [-1, 1]$

   For each $y \in [-1, 1]$ define $g(y) = x$ iff $y = \cos x$ for $x \in [0, \pi]$

$$g'(y) = \frac{1}{-\sqrt{1 - y^2}}$$

Lecture 33, Nov. 21

Exponential and Logarithmic Functions

33.1 Definition (a^x)**.** Let $a > 0$ We have

1. $a^0 = 1$
2. $a^n = a \cdot a \cdot a \cdots a$ if $n \in \mathbb{N}$
3. $a^{n/m} = \sqrt[m]{a^n}$
4. if $\alpha \in \mathbb{R}$, $\alpha > 0$, let $a^\alpha = \lim_{r_n \to \alpha} a^{r_n}$ where $r_n \subset \mathbb{Q}$, $r_n \geq 0$
5. If $\alpha < 0$, let $a^\alpha = \frac{1}{a^{-\alpha}}$

33.2 Theorem (Properties of a^x)**.**

1. *$a^{x+y} = a^x a^y$*
2. *$a^{x^y} = a^{xy}$*
3. *$f(x) = a^x$ is differentiable and $f'(x) = f'(0)f(x) = f'(0)a^x$*
4. *There exist a unique base "e" for which if $f(x) = e^x$ then $f'(0) = 1$.*

Note. The derivative of $f(x) = a^x$ at $x = 0$ varies continuously with a. It also increase with a.

33.3 Theorem (The function e^x)**.** *Properties*

1. *Domain $e^x = \mathbb{R}$*
2. *Range $e^x = \mathbb{R}^+ = y \in \mathbb{R} \mid y \geq 0$*
3. *e^x is strictly increasing and hence invertible.*
4. *$f'(x) = f'(0)f(x) = 1 \cdot e^x = e^x$ (Inverse for $f(x) = e^x$)*

33.4 Definition (Natural log)**.** We define $g(y) = \ln y : \mathbb{R}^+ \to \mathbb{R}$ by $g(y) = x$ if and only if $e^x = y$

From the Inverse Function Theorem,

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{e_0^x} = \frac{1}{y_0}$$

Thus if $g(y) = \ln y$ then $g'(y) = \frac{1}{y}$.

Note. If $a > 0$, then $a = e^{\ln a}$, then $a^x = e^{\ln a^x} = e^{x \ln a}$. If $h(x) = a^x$,then the Chain Rule shows that

$$h'(x) = \frac{d}{dx} e^{x \ln a} = \ln a e^{x \ln a} = \ln a \cdot a^x$$

In particular, $h'(0) = \ln a = \lim_{h \to 0} \dfrac{a^n - 1}{h}$

Note. If $a \neq 1$, $a > 0$, then $f(x) = a^x$ is 1-1 from $\mathbb{R}$ onto $\mathbb{R}^+$

33.5 Definition. $g(y) = \log_a y : \mathbb{R}^+ \to \mathbb{R}$ by $g(y) = x$ iff $a^x = y$.

$\log_a y = x \Leftrightarrow a^x = y \Rightarrow e^{x \ln a} = y \Rightarrow \ln(e^{x\ln a}) = \ln y$ and $x \ln a = \ln y$, then $x = \frac{\ln y}{\ln a}$.

Hence, $\log_a(y) = \dfrac{\ln y}{\ln a} \Rightarrow \dfrac{d}{dx}(\log_a(x) = \dfrac{d}{dx}(\dfrac{\ln x}{\ln a}) = \dfrac{1}{\ln ax}$

33.6 Example (On the final exam)**.** Let $f(x) = x^x = (e^{\ln x})^x = e^{x \ln x}$

Domain $f = \mathbb{R}^+$

Note. If $g(x) = x \ln x$

$g'(x) = \frac{x}{x} + \ln x = 1 + \ln x = 0 \Rightarrow x = \frac{1}{e}$

$$f'(x) = e^{x \ln x} \frac{d}{dx} x \ln x = (1 + \ln x)e^{x \ln x} = (1 + \ln x)x^x$$

33.7 Example.

$$g(x) = x^{\sin x} = e^{\ln x \sin x}$$

33.8 Example (Mean Value Theorem)**.** Question: Suppose that a car travels a distance of 110km in exactly 1hr. If the posted speed limit on the road is 100km/h. Can you prove that the car was speeding at some point.

Lecture 34, Oct. 23

34.1 Theorem (Rolle's Theorem)**.** *Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Assume that $f(a) = f(b)$. Then there exists $c \in (a, b)$ with $f'(c) = 0$.*

Proof. If $f(x)$ is constant on $[a, b]$, then if $c \in (a, b)$, then $f'(c) = 0$. If $f(x)$ is not constant on $[a, b]$, then $f(x)$ attains either its maximum or its minimum on some point $c \in (a, b)$. In either case, $f'(c) = 0$. ☐

34.2 Theorem (Mean Value Theorem)**.** *Assume that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ with*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Let $h(x) = f(x) - g(x)$. Then $h(a) = h(b) = 0$. Since $h(x)$ is continuous on $[a, b]$ and differentiable on (a, b) with $h(a) = h(b)$, by Rolle's Theorem, there exists $c \in (a, b)$ with

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Thus

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

☐

34.3 Proposition. *Let $f(x)$ be continuous on an interval I. Assume that $f'(x) = 0$ for each $x \in I$. Then there exists $C \in \mathbb{R}$ such that $f(x) = C$ for all $x \in I$.*

Proof. Let $x_0 \in I$. Let $f(x_0) = C$.Let $x \in I$, $x \neq x_0$. Then the MVT holds on the interval between x_0 and x. There exists $d \in I$ between x_0 and x with

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(d) = 0$$

Thus $f(x) = f(x_0) = C$. ☐

34.4 Definition (Antiderivatives)**.** Given a function $f(x)$ we say that $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Note. Suppose $F(x), G(x)$ are antiderivatives of $f(x)$. Then $F'(x) = f(x) = G'(x)$. Let $H(x) = F(x) - G(x)$. Then we have $H'(x) = 0$.

For any $f(x)$, if $F(x)$ is antiderivative of $f(x)$, then all antiderivatives are of the form $G(x) = F(x) + C$ for some C.

34.5 Theorem (Increasing Function Theorem)**.** *Assume that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) with $f'(x) \geq 0$ ($f'(x) > 0$) for all $x \in (a, b)$, then $f(x)$ is (strictly) increasing on (a, b).*

Proof. Let $x < y \in [a, b]$. By the MVT, there exists $x < c < y$ with

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

Since $f'(c) \geq 0$, we get $f(y) \geq f(x)$. □

Lecture 35, Nov. 24

35.1 Theorem. *If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) with $m \le f'(x) \le M$ on (a,b), then for each $x \in [a,b]$ we have*

$$f(a) + m(x-a) \le f(x) \le f(a) + M(x-a)$$

Proof. Pick $x \in (a,b]$. Then the Mean Value Theorem holds on $[a,x]$. So there exists a $c \in (a,x)$ with

$$\frac{f(x)-f(a)}{x-a} = f'(c).$$

Hence

$$m \le \frac{f(x)-f(a)}{x-a} \le M.$$

Then

$$f(a) + m(x-a) \le f(x) \le f(a) + M(x-a).$$

□

35.2 Theorem. *Assume that $f(x)$ is differentiable on an interval I with $|f'(x)| \le M$ for all $x \in I$. Then $f(x)$ is uniformly continuous on I.*

Proof. Let $\epsilon > 0$. Let $x, y \in I$ with $x \ne y$. Then by the Mean Value Theorem,

$$\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \le M$$

Then

$$|f(x)-f(y)| \le M\,|x-y|$$

Let $\delta = \epsilon/M$. If $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon$. Thus $f(x)$ is uniformly continuous. □

35.3 Question. Assume $f(x)$ is uniformly continuous on I and differentiable on I. Is $f'(x)$ bounded on I?

Lecture 36, Nov. 25

36.1 Theorem (Increasing Function Theorem). *Assume that $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) with $f'(x) \geq 0$ ($f'(x) > 0$) for all $x \in (a,b)$, then $f(x)$ is (strictly) increasing on (a,b).*

36.2 Theorem (Comparison Theorem). *Assume that f and g are differentiable on (a,b) and continuous on $[a,b]$. If $f(a) = g(a)$ and if $f'(x) < g'(x)$ for all $x \in (a,b)$, then $f(x) < g(x)$ for all $x \in (a,b]$.*

Classifying Critical Points

36.3 Theorem (First Derivative Test). *Assume that $f'(c) = 0$*

1. *Assume that there exists an open interval (a,b) containing c with $f'(x) \geq 0$ for all $a < x < c$ and $f'(x) \leq 0$ for all $c < x < b$, then $x = c$ is a local maximum.*
2. *Assume that there exists an open interval (a,b) containing c with $f'(x) \leq 0$ for all $a < x < c$ and $f'(x) \geq 0$ for all $c < x < b$, then $x = c$ is a local minimum.*

Proof. Let $a < x_0 < c$. Then Mean Value Theorem holds on $[x_0, c]$. There exists $d_1 \in (x_0, c)$ with

$$\frac{f(x_0) - f(c)}{x_0 - c} = f'(d_1) \geq 0.$$

Then $f(x_0) \leq f(c)$ since $x_0 - c < 0$. Similarly we prove the other parts of the theorem. □

36.4 Theorem (Second Derivative Test). *Assume that $f'(c) = 0$ and that $f''(x)$ is continuous at $x = c$.*

1. *If $f''(c) > 0$, then $x = c$ is a local minimum*
2. *If $f''(c) < 0$, then $x = c$ is a local maximum*

Proof. Assume that $f'(c) = 0$ and that $f''(x)$ is continuous at $x = c$.

1. Assume $f''(x) > 0$. Since $f''(x)$ is continuous at $x = c$, there is an open interval $(c-\delta, c+\delta)$ on which $f''(x) > 0$. Hence $f'(x)$ is strictly increasing on $(c-\delta, c+\delta)$. But $f'(c) < 0$, then $f'(x) < 0$ on $(c-\delta, c)$ and $f'(c) > 0$ on $(c, c+\delta)$. Then we apply the First Derivative Test.

□

36.5 Definition (Concavity). We say that a function $f(x)$ which is continuous on an interval I is concave up on I if for every $a < b$, $a, b \in I$, we have

$$h(x) = f(a) + \frac{f(b) - f(a)}{b - a} - f(x) \geq 0 \text{ on } (a,b)$$

We say that a function $f(x)$ is concave down on I if for every $a < b$, $a, b \in I$, we have

$$h(x) = f(a) + \frac{f(b) - f(a)}{b - a} - f(x) \leq 0 \text{ on } (a,b)$$

36.6 Theorem (Concavity Theorem).

1. *Assume that $f''(x) > 0$ for all $x \in I$ then $f(x)$ is concave up on I.*
2. *Assume that $f''(x) < 0$ for all $x \in I$ then $f(x)$ is concave down on I.*

Lecture 37, Nov. 28

37.1 Definition (Concavity)**.** We say that a function $f(x)$ which is continuous on an interval I is concave up on I if for every $a < b$, $a, b \in I$, we have

$$h(x) = f(a) + \frac{f(b) - f(a)}{b - a} - f(x) \geq 0 \text{ on } (a, b)$$

We say that a function $f(x)$ is concave down on I if for every $a < b$, $a, b \in I$, we have

$$h(x) = f(a) + \frac{f(b) - f(a)}{b - a} - f(x) \leq 0 \text{ on } (a, b)$$

37.2 Theorem (Concavity Theorem)**.**

1. *Assume that $f''(x) > 0$ for all $x \in I$ then $f(x)$ is concave up on I.*
2. *Assume that $f''(x) < 0$ for all $x \in I$ then $f(x)$ is concave down on I.*

Proof. Assume that $f''(x) \geq 0$ on I. Let $a < b$, $a, b \in I$. Consider $h(x) = f(x) - g(x)$ where $g(x) = f(a) + (f(b) - f(a))(x - a)/(b - a)$. Since $h(a) = h(b) = 0$, by Rolle's Theorem there exists a $c \in (a, b)$ with $h'(c) = 0$. Moreover, $h''(x) = f''(x) > 0$ on (a, b). Then $f'(x)$ is increasing on (a, b). Hence $h'(x) < 0$ on (a, c) and $h'(x) > 0$ on (c, b). Hence c is a local minimum. Moreover this is the only point in (a, b) with $h'(c) = 0$. □

37.3 Theorem. *Assume that $f\colon I \to \mathbb{R}$ is such that $f''(x) > 0$ on I. Then $f(x)$ is concave upwards on I. If $f''(x) < 0$ on I. Then $f(x)$ is concave downwards on I.*

37.4 Definition. Assume that $f(x)$ is continuous on (a, b) with $c \in (a, b)$. We say that c is an inflection point for $f(x)$ if there exist $\lambda > 0$ such that either i) $f(x)$ concave up on $(c - \lambda, c)$ and concave down on $(c, c + \lambda)$ or ii) $f(x)$ concave down on $(c - \lambda, c)$ and concave up on $(c, c + \lambda)$

37.5 Proposition. *Assume that $f''(x)$ exists on (a, b). If $c \in (a, b)$ is an inflection point, then $f''(c) = 0$.*

37.6 Theorem (Cauchy's Mean Value Theorem)**.** *Assume that $f(x)$ and $g(x)$ are continuous on $[a, b]$, differentiable on (a, b) with $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ with*

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(c)}{g'(c)}$$

Proof. Observe that $g'(x) \neq 0$ on (a, b) so $g(b) \neq g(a)$. Let

$$h(x) = \frac{f(b) - f(a)}{g(b) - g(a)}(f(x) - g(a)) - (f(x) - f(a))$$

Then since $h(a) = h(b) = 0$, by Rolle's Theorem there exists $c \in (a, b)$ with

$$h'(c) = 0 = \frac{f(b) - f(a)}{g(b) - g(a)}(f(x) - g(a))g'(c) - f'(c)$$

then

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(c)}{g'(c)}$$

□

37.7 Definition. Assume that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. We call $\lim_{x\to a} \frac{f(x)}{g(x)}$ on indeterminate form of type $\frac{0}{0}$ If $\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to 0} g(x)$. Then $\lim_{x\to a} \frac{f(x)}{g(x)}$ is type $\frac{\infty}{\infty}$

37.8 Theorem (L'Hospital's Rule Version 0/0)**.** *Assume that $f(x)$ and $g(x)$ are differentiable on (a, b) where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$. Assume also that $g(x) \neq 0$, $g'(x) \neq 0$ on (a, b). Assume that $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$, then*

1. *if* $\displaystyle\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, *then* $\displaystyle\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.
2. *if* $\displaystyle\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \pm\infty$, *then* $\displaystyle\lim_{x\to a^+} \frac{f(x)}{g(x)} = \pm\infty$.

Assume that $\lim_{x\to b^-} f(x) = 0 = \lim_{x\to b^-} g(x)$, *then*

3 *if* $\displaystyle\lim_{x\to b^-} \frac{f'(x)}{g'(x)} = L$, *then* $\displaystyle\lim_{x\to b^-} \frac{f(x)}{g(x)} = L$.

4 *if* $\displaystyle\lim_{x\to b^-} \frac{f'(x)}{g'(x)} = \pm\infty$, *then* $\displaystyle\lim_{x\to b^-} \frac{f(x)}{g(x)} = \pm\infty$.

Proof. Assume that

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$$

Let $\epsilon > 0$. There exists a β such that if $a < x < b$, then

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\epsilon}{2}$$

Let $a < \alpha < \beta_1 < \beta$. Then by the Cauchy Mean Value Theorem

$$\left|\frac{f(\beta_1) - f(\alpha)}{g(\beta_1) - g(\alpha)} - L\right| = \left|\frac{f'(c_{\alpha\beta_1})}{g'(c_{\alpha\beta_1})} - L\right|$$

Hence

$$\left|\frac{f(\beta_1)}{g(\beta_1)} - L\right| = \lim_{\alpha\to a^+} \left|\frac{f(\beta_1) - f(\alpha)}{g(\beta_1) - g(\alpha)} - L\right| \leq \frac{\epsilon}{2} < \epsilon$$

$\square$

Lecture 38, Nov. 30

38.1 Example.

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

38.2 Example.

$$\lim_{x \to 0} \frac{e^x - \cos x}{x}$$

38.3 Example.

$$\lim_{x \to 0} \frac{x}{e^x}$$

38.4 Example.

$$\lim_{x \to 0^+} x \ln x$$

38.5 Example.

$$\lim_{x \to 0^+} x^x$$

38.6 Example.

$$\lim_{x \to 0^+} x^{\sin x}$$

38.7 Definition. Recall: If $f(x)$ is differentiable at $x = a$, then $L_a^f(x) = f(a) + f'(a)(x - a)$ is the unique degree 1 (or less) polynomial with 1) $L_a^f(a) = f(a)$ 2) $L_a^{f'(a)} = f'(a)$

Question: Assume that $f''(a)$ exists, does there exist a polynomial $p(x) = a_0 + a_1(x - a) + a_2(x - a)^2$ with $p(a) = f(a)$, $p'(a) = f'(a)$, and $p''(a) = f''(a)$?

Note:

$$\begin{aligned}
&p(a) = a_0, p'(x) = a_1 + 2a_2(x - a) \\
&p'(a) = a_1 \Rightarrow Le + a_1 = f'(a) \\
&p''(x) = 2a_2 \Rightarrow 2a_2 = f''(a) \\
&p(x) = f(a) + f'(a)(x - a) + f''(a)(x - a)^2 \\
&p(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2
\end{aligned}$$

Question: Assume that $f_{(a)}^{(n)}$ exists. Then is there a polynomial of the form $P_{n,a}(x) = a_0 + a_1(x - a) + ... + a_n(x - a)^n$ where $P_{n,0}^{(k)}(a) = f_{(a)}^{(k)}$ for $k = 0, 1, 2, ..., n$?

The answer is YES, in fact (we define $0! = 1$):

$$\begin{aligned}
P_{n,a}(x) &= \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x - a)^n \\
P_{n,a}(x) &= \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k
\end{aligned}$$

38.8 Definition (n-th degree Taylor Polynomial)**.** Given a function $f(x)$ with $f^{(n)}(a)$. We define the n-th degree Taylor Polynomial for $f(x)$ centered at $x = a$ to be

$$P_{n,a}(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$P_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Observation:

1. $P_{0,a}(x) = f(a)$
2. $P_{1,a}(x) = L_a^f(x)$

38.9 Example. $f(x) = e^x, a = 0$

$$f'(x) = e^x = f''(x) = f'''(x) = ... = f^{(n)}(x)$$
$$f^k(0) = e^0 = 1$$

$$P_{0,0}(x) = e^0 = 1$$
$$P_{1,0}(x) = f(0) + f'(0)x = 1 + x$$
$$P_{2,0}(x) = 1 + x + \frac{f''(a)}{2!}x^2 = 1 + x + \frac{x^2}{2!}$$
$$P_{n,0}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^n}{n!}$$

Strategy: If $x \approxeq a$, $f(x) \approxeq P_{n,0}(x)$ (Hope: As n increases the approximation is more accurate)

Question: Can we quantify the error? i.e. How big is $f'(x) - P_{n,0}(x)$?

38.10 Definition.

$$R_{n,a}(x) = f(x) - P_{n,a}(x)$$

is the error in using the Taylor Polynomial to approximate $f(x)$ near $x = a$.

38.11 Theorem (Taylor's Theorem)**.** *Assume that $f^{(n+1)}(x)$ exists in an open interval I containing $x = a$ For each $x \in I (x \neq a)$ there exist c_x strictly between x and a, such that*

$$R_{n,a}(x) = f(x) - P_{n,a}(x) = \frac{f^{(x+1)}(x)}{(n+1)!}(x-a)^{x+1}$$

38.12 Example.

$$n = 1$$
$$R_{1,a}(x) = f(x) - P_{1,a}(x)$$
$$= f(x) - L_a^f(x)$$
$$= \frac{f''(x)}{2!}(x-a)^2$$

Lecture 39, Dec. 1

Proof of Taylor's Theorem. If $x \neq a$, let M be such that $R_{n,a} = f(x) = P_{n,a} \inf = M(x-a)^{n+1}$.

Let $F(t) = f(t) + f'(t) + f''(t)(x-t) + ...$

$F'(t) = f'(t) + (-f'(t) + f''(t)(x-a)) + ...$

$F'(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n - M(n+1)(x-t)^n$

$F'(c) = 0 \Rightarrow \frac{f^{(n+1)}}{n!} - M(n+1)(x-t)^n = M = \frac{f^{(n+1)}(c)}{(n+1)!}$ (Some stuff missing here) $\square$

Observation:

1. When $n = 0$, Taylor Thm is the MVT.
2. When $n = 1$, $P_{1,a}(x) = L_a^f(a) \Rightarrow \left|f(x) - L_a^f(x)\right| = \frac{f''(a)}{2!}(x-a)^2$

39.1 Example ($\sin x$)**.** Same on the modulo on Learn.

Some stuff missing here.

Lecture 40, Dec. 2

40.1 Example. Let

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}$$

Solution. From Taylor's Theorem we get

$$|\sin h - h| \leq \frac{|h^3|}{6}$$

Then

$$\left|\frac{\sin h}{h} - 1\right| \leq \frac{|h^3|}{6}$$

Then

$$\left|\frac{\frac{\sin h}{h} - 1}{h} - 0\right| \leq \frac{|h|}{6}$$

By Squeeze Theorem

$$\lim_{x\to 0} \frac{\frac{\sin h}{h} - 1}{h} - 0 = f'(0) = 0$$

40.2 Theorem (Approximation Theorem)**.** *Assume that there exists a* $\delta > 0$ *such that* $\left|f^{(n+1)}(x)\right| \leq M$ *for all* $x \in (a-\delta, a+\delta)$*, then for each* $x \in (a-\delta, a+\delta)$ *we have*

$$|f(x) - P_{n,a}(x)| \leq \frac{M}{(n+1)!}\left|(x-a)^{n+1}\right|$$

40.3 Definition (Big-O notation)**.** Let $a \in \mathbb{R}$. Given f, g we say that $f(x) = O(g(x))$ as x approaches a if there exists $0 < \delta \leq 1$ with

$$|f(x)| \leq M\,|f(x)| \text{ for all } x \in (a-\delta, a+\delta)$$

except possibly at $x = a$.

40.4 Theorem. *If There exists a* $0 < \delta \leq 1$ *such that* $f^{(n+1)}(x)$ *is continuous on* $[-\delta, \delta]$*, then*

$$f(x) - P_{n,a}(x) = O(x^{n+1})$$

and we write $f(x) = P_{n,a}(x) + O(x^{n+1})$.

Proof. Since $f^{(n+1)}(x)$ is continuous on $[-\delta, \delta]$, the Extreme Value Theorem show that there exists M with $\left|f^{(n+1)}(x)\right| \leq M$ for all $x \in [-\delta, \delta]$. Hence by the Approximation Theorem,

$$|f(x) - P_{n,a}(x)| \leq \frac{M}{(n+1)!}\left|x^{n+1}\right|$$

Then $f(x) = P_{n,a}(x) + O(x^{n+1})$. □

40.5 Theorem (Arithmetic Rules for Big-O)**.** *Assume that* $f = O(x^n)$*,* $g = O(x^m)$*.*

1. $cf(x) = O(x^n)$

2. $f(x) + g(x) = O(x^{\min(m,n)})$

3. $f(x) \cdot g(x) = O(x^{m+n})$

4. $x^k \cdot f(x) = O(x^{n+k})$

5. $(f(x))^k = O(x^{nk})$

Proof. On $[-\delta, \delta]$, $|f(x)| \le M_1 |x^n|$, $|g(x)| \le M_2 |x^n|$ Then

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le M_1 |x^n| + M_2 |x^m| \le M_1 \left|x^{\min(m,n)}\right| + M_2 \left|x^{\min(m,n)}\right| = (M_1 + M_2) \left|x^{\min(m,n)}\right|$$

□

40.6 Lemma. *Let $p(x) = a_0 + a_1 x + \cdots a_n x^n$. Assume that $p(x) = O(x^{n+1})$, then $p(x) = 0$.*

Proof. Prove by induction. □

40.7 Theorem. *Assume that $f^{(n+1)}(x)$ is continuous on $[-\delta, \delta]$. If $p(x) = a_0 + a_1 x + \cdots a_n x^n$ is such that $f(x) = p(x) + O(x^{n+1})$, then $p(x) = P_{n,0}(x)$.*

Proof.

$$\begin{aligned} p(x) - P_{n,a}(x) =& (p(x) - f(x)) + (f(x) - P_{n,0}(x)) \\ =& O(x^{m+1}) + O(x^{m+1}) \\ =& O(x^{m+1}) \end{aligned}$$

Then by the lemma, we have $p(x) - P_{n,a}(x) = 0$. □

Tutorial 1, Sept. 15

Office Hours M 9:30-10:45
W 1:45-2:30
Th 2:30-4:00
F 9:30-10:45

Functions

41.1 Definition. A function is a rule that assigns to each element x in a set X a single value y in a set Y.

Notation.

$$f\colon X \to Y$$

where X is the domain of f and Y -is the codomain of f.

41.2 Example. $X = \mathbb{R}$ and $Y = \mathbb{R}$,

$$y = f(x) = x^2$$

41.3 Definition. Given $f\colon X \to Y$, the range of f is

$$ran(f) = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$$

We say that $f\colon X \to Y$ is onto if $ran(f) = codomain(f)$.

41.4 Example. $f(x) = x^3$, $f\colon \mathbb{R} \to \mathbb{R}$. f is onto.

$f\colon X \to Y$ is 1-1 if whenever $x_1, x_2 \in X$ with $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

If f is 1-1 and onto, for each $y \in Y$ we define

$$g(y) = x \iff y = f(x)$$

This gives us a function $g\colon Y \to X$ which is called the inverse of f and denoted by f^{-1}

Properties Suppose $f\colon X \to Y$, $g\colon Y \to Z$, we get

$$g \circ f = g(f(x))$$

and

$$g \circ f\colon X \to Z$$

Suppose that $f\colon X \to Y$ is 1-1 and onto with inverse g,

$$g \circ f(x) = x$$

Tutorial 2, Sept. 22

Additional Office Hours

- Tuesday 1:30-2:30 MC 5413
- Friday 2:30-3:30 MC 5417

Tutorial 3, Oct. 14

a) WA2 due Monday 2:30

b) Midterm on Monday Oct. 24

43.1 Definition. A sequence $\{a_n\}$ is Cauchy if for every $\epsilon > 0$ there exists an $N_0 in \mathbb{N}$ so that if $n, m \geq N_0$, then

$$|a_n - a_m| < \epsilon.$$

43.2 Question. Let $\{a_n\}$ be such that

$$\lim_{n\to\infty} a_{n+1} - a_n = 0.$$

Is $\{a_n\}$ Cauchy?

Solution. No. e.g.

$$a_n = \sum_{k=1}^{n} \frac{1}{k}$$